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All contributions intended for publication in *Mathematical Tables and Other Aids to Computation* and all books for review should be addressed to C. B. Tompkins, Department of Mathematics, University of California, Los Angeles 24, California. The author should mention the name of an appropriate editor for his paper if this is convenient.

Estimation of the Relaxation Factor
for Small Mesh Size

Recently Frankel [1] and Young [3, 4] have developed a method of successive overrelaxation for the solution of the difference equation analogue of the Dirichlet problem for a linear elliptic partial differential equation. The purpose of this note is to present a simple asymptotic theory of the overrelaxation process which depends on the assumption that the mesh size h approaches zero, but which describes adequately the practical aspects of the earlier work. Thus the results discussed are not essentially new, but it is hoped that our approach to them will provide an easy insight into the overrelaxation scheme for those interested in its application to specific examples.

We consider the Laplace difference equations for an unknown function u of two independent variables in a region D covered by net points spaced h units apart. In a standard notation, we use subscripts i and j to refer to the location of net points in our grid and we use the superscript n to indicate steps in the relaxation process, so that, in its simplest form, the method of successive overrelaxation can be described by the equation

$$(1) \qquad 4(u_{i,j}^{n+1} - u_{i,j}^n) = r(u_{i-1,j}^{n+1} + u_{i,j-1}^{n+1} + u_{i+1,j}^n + u_{i,j+1}^n - 4u_{i,j}^n),$$

where r is the relaxation factor. We express r in the form

$$(2) \qquad r = \frac{2}{1 + Ch}$$

for any positive value of the constant C, and we rearrange (1) to obtain

$$(3) \qquad \frac{u_{i-1,j}^n + u_{i,j-1}^n + u_{i+1,j}^n + u_{i,j+1}^n - 4u_{i,j}^n}{h^2} = \frac{u_{i,j}^{n+1} - u_{i,j}^n - u_{i-1,j}^{n+1} + u_{i-1,j}^n}{h^2}$$
$$+ \frac{u_{i,j}^{n+1} - u_{i,j}^n - u_{i,j-1}^{n+1} + u_{i,j-1}^n}{h^2} + 2C\frac{u_{i,j}^{n+1} - u_{i,j}^n}{h}.$$

Using the familiar idea that the index n refers to a new time variable, t, and that it indicates the location of new net points spaced at time intervals equal to the original mesh size h, we recognize that (3) is the difference analogue of the hyperbolic partial differential equation

$$(4) \qquad u_{xx} + u_{yy} = u_{xt} + u_{yt} + 2C u_t.$$

Thus for small values of h the convergence of the iterative scheme (1) can be investigated by an analysis of the decay of time-dependent terms in the solution of (4).

The substitution $s = t + x/2 + y/2$ brings (4) into the canonical form

$$(5) \qquad u_{zz} + u_{yy} - \tfrac{1}{2}u_{ss} - 2C\,u_s = 0.$$

For a fixed set of boundary conditions, the method of separation of variables yields the representation

$$(6) \qquad u = U_0(x, y) + \sum_{m=1}^{\infty} [a_m \exp(-p_m s) + b_m \exp(-q_m s)]U_m(x, y)$$

for the solution u of (5), where U_0 is the steady-state solution, where the a_m and b_m are Fourier coefficients, where

$$(7) \qquad p_m = 2C - (4C^2 - 2k_m^2)^{\frac{1}{2}}, \quad q_m = 2C + (4C^2 - 2k_m^2)^{\frac{1}{2}},$$

and where the U_m and k_m^2 are the eigenfunctions and eigenvalues of the problem

$$(8) \qquad \Delta U_m + k_m^2 U_m = 0$$

with homogeneous boundary conditions. The exponent

$$(9) \qquad p = \mathrm{Re}\,[p_1] = \mathrm{Re}\,[2C - (4C^2 - 2k_1^2)^{\frac{1}{2}}]$$

corresponding to the lowest eigenvalue k_1^2 governs the rate of decay of the terms on the right in (6) with increasing time t. Thus the error E in the solution of the Laplace difference equations obtained through N iterations of the overrelaxation scheme (1) has the order of magnitude

$$(10) \qquad E = O(e^{-pNh}),$$

since it is approximated by the infinite series used in the representation (6) for time $t = Nh$. It follows that we must iterate $N = O(1/h)$ times to arrive at a desired degree of accuracy in the solution.

By (9), the choice of the positive constant C which maximizes p and therefore yields the most rapid convergence is clearly $C = k_1/2^{\frac{1}{2}}$, and an underestimate of this best value is less damaging than an overestimate (cf. [3]). Indeed, the radical in the expression for p is pure imaginary and can be omitted when $C < k_1/2^{\frac{1}{2}}$, whereas this same radical causes a sharp decline in the value of p when C is taken slightly larger than $k_1/2^{\frac{1}{2}}$. A simple lower bound on the principal frequency k_1 for the case of Dirichlet boundary conditions is given by Pólya and Szegö [2]. If A denotes the area of the region D, they show that

$$(11) \qquad k_1 A^{\frac{1}{2}} \geq k\pi^{\frac{1}{2}},$$

where $k = 2.405$ denotes the first root of the Bessel function of the first kind of order zero. Thus a good approximate formula for the relaxation factor r is

$$(12) \qquad r = \frac{2}{1 + (\pi/2A)^{\frac{1}{2}}kh} = \frac{2}{1 + 3.014h/A^{\frac{1}{2}}}.$$

Numerical examples worked out by Young [4] indicate that when (12) is used instead of the exact best value of r, the additional iterations required to achieve equivalent accuracy will usually not exceed 20 per cent of the number needed with the optimal r.

Our approach shows that the method of successive overrelaxation applies to the nine-point Laplace difference equation

$$(13) \quad 4u_{i-1,j} + 4u_{i,j-1} + 4u_{i+1,j} + 4u_{i,j+1} + u_{i-1,j-1} + u_{i+1,j-1}$$
$$+ u_{i+1,j+1} + u_{i-1,j+1} - 20u_{i,j} = 0.$$

It is easily verified that a relaxation factor r of the form (2) leads in the case of (13) to the hyperbolic partial differential equation

$$(14) \qquad\qquad 3u_{xx} + 3u_{yy} = 2u_{xt} + 3u_{yt} + 5Cu_t,$$

similar to (4), and that the best choice for the parameter C here is $C = 13^{\frac{1}{2}}k_1/5$, while the corresponding value for the exponent p governing the rate of convergence of the iterations is $p = 6k_1/13^{\frac{1}{2}}$. Thus we see that these quantities are slightly larger for the nine-point equation than they are for the five-point equation, a result not inconsistent with the numerical data compiled by Young [4], which indicated that the optimal value of r decreases and the correspondingly smallest number of required iterations changes insignificantly when we pass from the five-point to the nine-point interpolation formula. The strong dependence of the latter formula for the Laplace operator on the particular partial differential equation involved appears to have obscured until now the explanation of this phenomenon. At all events, for the nine-point difference equation the asymptotic expression for the best value of the relaxation factor r is

$$(15) \qquad\qquad r = \frac{2}{1 + 13^{\frac{1}{2}}k_1h/5} \le \frac{2}{1 + 3.074h/A^{\frac{1}{2}}}.$$

Finally, our introduction of the second order hyperbolic differential equation (4) indicates an analogy between the successive overrelaxation scheme and the second order Richardson method, which is based on a more readily accessible hyperbolic equation (cf. [1]) and converges at a comparable speed. The advantage of successive overrelaxation is that it requires retention of less data and is therefore more adaptable for use on high speed computing machines.

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2. G. PÓLYA & G. SZEGÖ, *Isoperimetric Inequalities in Mathematical Physics*, Princeton Univ. Press, 1951.

3. DAVID YOUNG, "Iterative methods for solving partial difference equations of elliptic type," Amer. Math. Soc., *Trans.*. v. 75, 1954, p. 92–111.

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Uniqueness of the Projective Plane of Order Eight

1. Introduction. A finite projective plane with $n + 1$ points on each of its lines is said to be of order n. Such planes are known to exist whenever n is a prime or prime power, there being at least the Desarguesian plane coordinatized by the finite field with n elements [8]. For some time it has been known [9] that for order 9 there are some non-Desarguesian planes. It is an easy exercise to construct the planes of orders 2, 3, 4, and 5 and to verify that they are unique. The results of Bruck and Ryser [1] include as a special case the nonexistence of planes of order 6 although this special case had been shown earlier by Tarry [7]. The combined results of Pierce [5] and Hall [2] show the uniqueness of the plane of order 7. In this paper we show the uniqueness of the plane of order 8, thus showing that 9 is the smallest order of a non-Desarguesian plane.

The demonstration of the uniqueness of the plane of order 8 is based on Norton's complete list of Latin squares of order 7 [4], there being a total of 147 varieties of these, an omission in Norton's list having been found by Sade [6]. By a theoretical argument it is necessary to consider only 100 of these, and machine calculations, described in section 4 of this paper carried out on SWAC at the University of California, Los Angeles, went far enough so that it was not difficult to complete the work by hand. The completion of the search is given in section 5 of this paper.

2. Theoretical basis for calculation. Let A, B, C be the vertices of a triangle in a plane of order n. Call AB the line at infinity L_∞, AC the line $x = 0$, and BC the line $y = 0$. Label the $n - 1$ remaining lines through A as $x = 1$, $x = 2$, $\cdots$, $x = n - 1$ in any order and also the $n - 1$ remaining lines through B as $y = 1$, $\cdots$, $y = n - 1$ in any order. A point P not on L_∞ will then lie on a unique line $x = a$ and a unique line $y = b$. Then assign to P the coördinates (a, b).

The $n - 1$ lines through $C = (0, 0)$ apart from AC and BC will intersect each of $x = 1$, $\cdots$, $x = n - 1$ once and each of $y = 1$, $\cdots$, $y = n - 1$ once. Such a line L will intersect L_∞ in some infinite point and will also contain $(0, 0)$ and $n - 1$ points (i, j) where i and j take values 1 to $n - 1$. With L associate the permutation $\begin{pmatrix} 0, 1, & \cdots, n - 1 \\ 0, a_1, & \cdots, a_{n-1} \end{pmatrix}$ if $(0, 0)$, $(1, a_1)$, $\cdots$, $(n - 1, a_{n-1})$ are the finite points of L. The $n - 1$ different L's yield permutations:

$$
\begin{matrix}
a_{11}, & a_{12}, & \cdots, & a_{1,n-1} \\
a_{21}, & a_{22}, & \cdots, & a_{2,n-1} \\
\cdot & & & \cdot \\
\cdot & & & \cdot \\
\cdot & & & \cdot \\
a_{n-1,1}, & a_{n-1,2}, & \cdots, & a_{n-1,n-1}
\end{matrix}
$$

(2.1)

and the array (2.1) will form a Latin square of order $n - 1$. Since a line through C intersects each of $y = 1$, $\cdots$, $y = n - 1$ once, each row of (2.1) contains each of 1, $\cdots$, $n - 1$ once. Since a line $x = i$ intersects the $n - 1$ L's in different points the i-th column will contain each of $1, 2, \cdots, n - 1$ once. The three constraints, row, column, digit correspond respectively to a line through C, a line $x = i$ and

a line $y = j$. The interchanging of these three constraints corresponds therefore to an interchange of the roles of the vertices A, B, C in the construction of het Latin square. A permutation of the rows is merely a permutation of the order in which the lines through C are taken. A permutation of the columns corresponds to a relabeling of the lines $x = c$ through A and a substitution on the digits corresponds to a relabeling of the lines $y = c$ through B. Thus, these equivalent forms of Latin squares obtained by permuting rows, permuting columns, and substituting on digits, amount to no more than relabeling the lines of the three-net of pencils through the vertices of a triangle A B C excluding the sides. And an interchange of constraints corresponds to altering the roles of the vertices A, B, C in obtaining the square (2.1). It is immediate that any Latin square of order $n - 1$ may be used as an array (2.1) corresponding to a three-net of the pencils through three points A, B, C excluding the sides. It is not immediate nor even true that any Latin square yields a three-net which can be extended to a complete plane. (However, for countably infinite planes this and more is true.) [3].

Norton [4] has given a list of the Latin squares of order seven to within the equivalences of permuting rows, columns, substituting digits and interchanging the constraints. Norton lists 146 varieties. Sade [6] found an omission and verified that with this additional variety included the list is complete.

These 147 squares form the starting point in our search for planes of order 8. We seek to complete these by adding as many as possible additional lines not through A, B, or C. It is however sufficient to treat only 100 of the 147, effecting a saving of approximately one-third of the machine time. Norton lists his squares according to the number of intercalates which they contain, an intercalate being a sub-array of the type:

(2.2)

$$\begin{matrix} a & \cdots & b \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ b & \cdots & a. \end{matrix}$$

If the columns of the intercalate (2.2) are the i-th and j-th then the four points (i, a), (i, b), (j, a), and (j, b) are such that no three lie on a line and so they form a quadrilateral. The diagonal points of this quadrilateral are the points A, B, C. Conversely if A, B, C are the diagonal points of a quadrilateral these four points yield an intercalate in the array (2.1). Now in a plane of order 8 there are $73 \cdot 72 \cdot 64 \cdot 49 / 1 \cdot 2 \cdot 3 \cdot 4$ quadrilaterals and there are $73 \cdot 72 \cdot 64 / 1 \cdot 2 \cdot 3$ triangles. Now for some quadrilaterals the diagonal points may lie on a line. (Indeed in the Desarguesian plane of order 8 this is always the case.) Thus the triangles of a plane of order 8 are diagonal points of quadrilaterals at most $73 \cdot 72 \cdot 64 \cdot 49 / 1 \cdot 2 \cdot 3 \cdot 4$ times and as there are $73 \cdot 72 \cdot 64 / 1 \cdot 2 \cdot 3$ triangles there must be a triangle A, B, C which yields a square with at most $49/4$ intercalates and thus at most 12 intercalates. Hence in constructing a plane of order 8 we may start from a square of order 7 with at most 12 intercalates. Such squares are numbers 1 through 99 of Norton's list and the omission found by Sade.

3. Completion of squares to the full plane. The one hundred squares of order 7 used in the search for planes of order 8 could all be normalized so that the first

two lines read, adding an initial 0 for $C = (0, 0)$,

$$(3.1) \quad \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 2 & 3 & 4 & 5 & 6 & 7 & 1. \end{array}$$

It is sufficient to deal with the finite points (i, j) $i, j = 0, 1, \cdots, 7$ constructing an affine plane of order 8 since the completion to a projective plane by adding infinite points and L_∞ is trivial. The pencil through $(1, 1)$ will include the lines $x = 1, y = 1$ and the line 0 1 2 3 4 5 6 7 of (3.1). The remaining lines through $(1, 1)$ will go through the points $(2, 3)$, $(3, 4)$, $(4, 5)$, $(5, 6)$, and $(6, 7)$ of the second line of (3.1) and there will also be a line through $(1, 1)$ parallel to the second line of (3.1). These six lines will thus have the appearance:

$$(3.2) \quad \begin{array}{cccccccc} X & 1 & 3 & X & X & X & X & X \\ X & 1 & X & 4 & X & X & X & X \\ X & 1 & X & X & 5 & X & X & X \\ X & 1 & X & X & X & 6 & X & X \\ X & 1 & X & X & X & X & 7 & X \\ X & 1 & X & X & X & X & X & X. \end{array}$$

Here X's stand for digits to be filled in. Similarly when the six lines of (3.2) have been added to the lines given by one of the 100 squares there will be five more lines of the pencil through $(2, 3)$ whose appearance will be:

$$(3.3) \quad \begin{array}{cccccccc} X & X & 3 & X & 4 & X & X & X \\ X & X & 3 & X & X & 5 & X & X \\ X & X & 3 & X & X & X & 6 & X \\ X & X & 3 & X & X & X & X & 7 \\ X & X & 3 & X & X & X & X & X. \end{array}$$

Earlier calculations of the lines of (3.2) were made on the Engineering Research Associates 1101 Computer. For each of the 100 squares a number of possibilities for (3.2) was found, there being between 20 and 25 for each of the 100 squares on the average. In this search the 1101 was used to perform somewhat complicated sorting procedures. Of the 7! = 5040 permutations of the form X 1 X X X X X there are 1478 which are consistent with (3.1) in that they do not agree with either line of (3.1) in as many as two places. These are distributed as follows:

Type

$$\begin{array}{llr} & X\ 1\ 3\ X\ X\ X\ X\ X & - 135 \\ & X\ 1\ X\ 4\ X\ X\ X\ X & - 140 \\ & X\ 1\ X\ X\ 5\ X\ X\ X & - 141 \\ & X\ 1\ X\ X\ X\ 6\ X\ X & - 140 \\ & X\ 1\ X\ X\ X\ X\ 7\ X & - 135 \\ \text{Parallel to second line} & X\ 1\ X\ X\ X\ X\ X\ X & - 787. \end{array}$$

The set of 5040 permutations was easily built up on punched cards and the 1478 consistent cards were sorted from these. These 1478 permutations were taken as input for the ERA 1101 and were handled in two stages. In the first stage permutations were eliminated which were inconsistent with any one of the five remaining lines of a particular square and the consistent lines were punched as

output on tape. For each of the squares the number of consistent lines was approximately 115 to 120. This was a fast code, most of the time being taken up by punching outputs, and averaged about three minutes for each square. The second stage involved taking the 115 to 120 consistent lines for each square and forming the six lines (3.2) from these in every possible way. This was relatively slow taking about half an hour for each square and yielding, as remarked above, about twenty solutions for each square. In every case tried, except for the Desarguesian plane, only three or four further lines could be added. But working by hand it took at least an hour to test the construction of further lines given a square and the lines of (3.2).

4. Description of the SWAC code. Making one hundred separate runs SWAC took each 7×7 square with no more than twelve intercalates and constructed the six lines of (3.2) and the five lines of (3.3), punching them as output whenever all eleven were found. The method employed was to try, for each of the X's in (3.2) and (3.3), all possibilities $0, 1, \cdots, 7$ compatible with previously assigned X's. To specify the criteria of compatibility let $P_i(j)$ be the j-th element of the i-th row of the 7×7 square augmented by an initial column of zeros, and $P_i'(j)$ the similar element of (3.2).

For any (i, j) referring to an X in (3.2) all $P_k'(l)$ with $k < i$ or with $k = i$ and $l < j$ will be assumed already assigned. The restrictions on $P_i'(j)$ are then

(a) $P_i'(j) \neq 1$,
(b) $P_i'(j) \neq i + 2$ if $i < 6$,
(c) $P_i'(j) \neq P_i'(l)$ for any $l < j$,
(d) $P_i'(j) \neq P_k'(j)$ for any $k < i$,
(e) If for any $l < j$ or $l = 2$ or $l = i + 2$ we have $P_i'(l) = P_k(l)$, then $P_i'(j) \neq P_k(j)$.

The first three conditions insure that each row of (3.2) is a permutation. The last two prevent any such row from agreeing in two positions with any previous row of (3.2) or any row of the 7×7 square.

For an element $P_i''(j)$ of (3.3) we have conditions like these plus an additional condition similar to (e) but with P replaced by P'. In the following discussion we shall consider only the assignment of the $P_i'(j)$; the case of the $P_i''(j)$ is more complicated due to the extra condition but is basically the same.

To apply these criteria we construct the set of allowable values of $P_i(j)$. If

$$S^* = \{1\} \cup \begin{Bmatrix} \{i + 2\} & \text{if} & i < 6 \\ \phi & \text{if} & i = 6 \end{Bmatrix} \cup \{P_i'(l) \mid l < j\},$$

then any $P_i'(j) \epsilon \bar{S}^*$ (the complement of S^*) will satisfy (a), (b), and (c). Similarly (d) and (e) will be satisfied if $P_i'(j) \epsilon \bar{S}_j$, where

$$S_j = \{P_k'(j) \mid k < i\} \cup \{P_k(j) \mid P_k(2) = 1 \text{ or } P_k(i + 2) = i + 2$$
$$\text{or } P_k(l) = P_i'(l) \text{ for some } l < j\}.$$

(Because of (3.1) the conditions $P_k(2) = 1$ and $P_k(i + 2) = i + 2$ imply $k = 1$ and $k = 2$ respectively. This convenient simplification is not essential.) Hence $S = \bar{S}^* \cap \bar{S}_j$ is the set of values of $P_i'(j)$ compatible with those already assigned.

To select a $P_i'(j)$ one computes S^*, S_j, and S. Suppose first that S is not

empty. Then $P_i'(j)$ is taken to be the smallest element of S, and one moves on to the next unfilled position in (3.2). If this position is in the same row S^* is replaced by $S^* \cup \{P_i'(j)\}$, j replaced by $j + 1$ (or $j + 2$ if $j = i + 1$), and the new S_j calculated. On moving to a new row i is increased by 1, j put equal to 1, and S^* set equal to $\{1, i + 2\}$ if $i < 6$ or $\{1\}$ if $i = 6$. When (3.2) is completely filled one starts a similar process in (3.3); when this is filled one records all the $P_i'(j)$ and $P_i''(j)$.

Suppose now that for some (i, j) the corresponding S is empty. This means that the previously assigned $P_i'(j)$'s must be changed. One therefore "backtracks," returning to the previously considered $P_i(j)$ by reducing j or, if necessary to move up to the preceding row, by reducing i and putting $j = 8$. In the former case one gets back the old S^* by removing the element $P_i'(j)$; in the latter one puts $S^* = \{0, 1, \cdots, 7\}$. The old S_j could be re-computed, or it could have been saved from the time of its previous use. Since the calculation of S_j is somewhat complicated, and since there is much back-tracking and advancing within a row but not much between rows, it was decided to save the S_j within each row but to re-compute them on back-tracking to another row.

After back-tracking and re-computing S one must of course avoid selecting from S the same $P_i'(j)$ that had been used before. This is accomplished by removing from S all elements less than or equal to the $P_i'(j)$ that was last used. In this way the successive selections of $P_i'(j)$ for a given (i, j) are strictly increasing until all possibilities are exhausted and one backtracks still farther.

After a solution is recorded the process is continued by the standard backtracking procedure, just as if one had proceeded to a next step and found $S = \phi$. The whole process stops when all possibilities for $P_1'(1)$ have been exhausted.

In applying a routine of this sort it is obviously advantageous to eliminate many possibilities early in the process. In particular, in each row of (3.2) it is helpful if S_j is as large as possible for small j. In the set-up as described

$$S_1 = \{P_k'(1) \mid k < i\} \cup \{0\}.$$

However if we make a cyclic permutation of columns so that the first element of each row becomes the last we get

$$S_1 = \{P_k'(1) \mid k < i\} \cup \{1, 2\}.$$

This slightly better situation was used in the coding.

In coding this process for SWAC the significant factor was the small (256 word) high-speed memory (HSM). The program had to be stored on the magnetic drum and appropriate parts transferred to the HSM when needed. Fortunately it was just possible to squeeze into the HSM the whole procedure for handling the $P_i'(j)$ and also for the $P_i''(j)$. The delays due to drum transfers entered only in the transfer from one of these routines to the other, and in the negligible number of input and output processes. Elements of a set were represented by positions within a word; i.e., the subset $\{a_1, a_2, \cdots\}$ of $\{0, 1, \cdots, 7\}$ was represented by the number $\sum 2^{-a_i-1}$. This representation makes it easy to code the set operations of union, intersection, and picking the smallest element.

The general flow of the code may be described by considering fourteen subroutines as follows:

Routine	Possible Predecessors	Operation Performed	Criterion for Successor	Successor Yes	No
A	(G_1) (New run)	Input given square, catalogue it by element and position	—	B_1	B_1
B_1	A, D_1, F_1, G_2	Compute possible values available for new position	Are any values available?	C_1	E_1
C_1	B_1	Store first available value in proper position	—	D_1	D_1
D_1	C_1	Advance position	Is first set of lines complete?	B_2	B_1
E_1	B_1	Test whether backtrack is possible	Is a position open in which to backtrack?	F_1	G_1
F_1	E_1	Backtrack. Go back to nearest available position and recompute	—	B_1	B_1
G_1	E_1	Halt or input new square	—	(A)	(A)
B_2	D_1, D_2, F_2	As in B_1	As in B_1	C_2	E_2
C_2	B_2	As in C_1	—	D_2	D_2
D_2	C_2	As in D_1	Is second set of lines complete?	P	B_2
E_2	B_2	Test whether backtrack would carry back into first six lines formed	Is backtrack in last five lines possible?	F_2	G_2
F_2	E_2	Backtrack as in F_1	—	B_2	B_2
G_2	E_2	Backtrack to next to last position of first six lines	—	B_1	B_1
P	D_2	Punch 11 lines	—	E_2	E_2
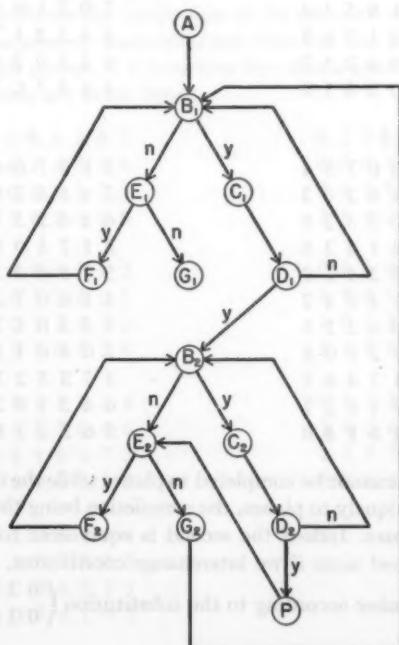
5. Results of calculations. For only one of the hundred 7×7 squares was it possible to construct the six lines of (3.2) and the five of (3.3). This was Square Number 1:

$$
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 2 & 3 & 4 & 5 & 6 & 7 & 1 \\
0 & 3 & 4 & 5 & 6 & 7 & 1 & 2 \\
0 & 4 & 5 & 6 & 7 & 1 & 2 & 3 \\
0 & 5 & 6 & 7 & 1 & 2 & 3 & 4 \\
0 & 6 & 7 & 1 & 2 & 3 & 4 & 5 \\
0 & 7 & 1 & 2 & 3 & 4 & 5 & 6,
\end{array}
$$

and four sets of eleven lines were found for this. The computing time for a square varied from eight to fifteen minutes, averaging about twelve minutes.

The four ways of adding the eleven lines to Square Number 1 are:

1	2
2 1 3 5 7 0 4 6	2 1 3 5 7 0 4 6
5 1 6 4 3 7 2 0	3 1 7 4 6 2 5 0
6 1 7 0 5 4 3 2	7 1 0 6 5 4 3 2
4 1 0 7 2 6 5 3	5 1 4 7 2 6 0 3
3 1 4 6 0 2 7 5	6 1 5 2 0 3 7 4
7 1 5 2 6 3 0 4	4 1 6 0 3 7 2 5
5 6 3 7 4 1 0 2	1 7 3 6 4 2 0 5
6 7 3 1 0 5 2 4	7 0 3 1 6 5 2 4
4 0 3 2 1 7 6 5	5 4 3 2 1 7 6 0
1 4 3 0 6 2 5 7	6 5 3 0 2 4 1 7
7 5 3 6 2 4 1 0	4 6 3 7 0 1 5 2

3	4
2 1 3 6 0 7 5 4	2 1 3 7 6 4 0 5
5 1 7 4 6 2 0 3	7 1 5 4 2 0 3 6
7 1 4 0 5 3 2 6	6 1 0 2 5 7 4 3
4 1 0 2 7 6 3 5	3 1 7 5 0 6 2 4
3 1 6 5 2 4 7 0	5 1 4 6 3 2 7 0
6 1 5 7 3 0 4 2	4 1 6 0 7 3 5 2
6 7 3 0 4 2 1 5	1 5 3 0 4 7 2 6
1 4 3 7 2 5 0 6	7 0 3 6 1 5 4 2
5 0 3 1 7 4 6 2	4 7 3 5 2 1 6 0
4 6 3 5 1 0 2 7	6 4 3 1 0 2 5 7
7 5 3 2 6 1 4 0	5 6 3 2 7 0 1 4

Of these the first two cannot be completed to planes while the third and fourth can both be completed uniquely to planes, the completion being the unique Desarguesian plane in both cases. Indeed the second is equivalent to the first so far as completion is concerned since if we interchange coordinates, replacing (x, y) by (y, x) and then renumber according to the substitution $\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 6 & 4 & 2 & 7 & 5 & 3 \end{pmatrix}$ the

first six lines of the second case become the first six lines of the first case, while this change leaves the lines of Square Number 1 unchanged. Thus if either of the first two can be completed then both can. But in fact neither can. In the first case no line $X\ X\ X\ 6\ X\ 7\ X\ X$ can be added while in the second no line $X\ X\ 7\ X\ X\ 6\ X\ X$ can be added.

In the fourth case if we interchange coordinates, replacing (x, y) by (y, x) and then renumber all coordinates according to the substitution $\begin{pmatrix} 0\ 1\ 2\ 3\ 4\ 5\ 6\ 7 \\ 0\ 1\ 4\ 7\ 3\ 6\ 2\ 5 \end{pmatrix}$, Square Number 1 is unaltered and the first six lines of the fourth case become the first six lines of the third case, whence these two cases must lead to the same completions. The completion is unique and is the Desarguesian plane. In the third case this may be written:

$$
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 0 & 6 & 4 & 3 & 7 & 2 & 5 \\
2 & 6 & 0 & 7 & 5 & 4 & 1 & 3 \\
3 & 4 & 7 & 0 & 1 & 6 & 5 & 2 \\
4 & 3 & 5 & 1 & 0 & 2 & 7 & 6 \\
5 & 7 & 4 & 6 & 2 & 0 & 3 & 1 \\
6 & 2 & 1 & 5 & 7 & 3 & 0 & 4 \\
7 & 5 & 3 & 2 & 6 & 1 & 4 & 0,
\end{array}
$$

along with six other 8×8 squares whose top rows are the remaining lines of Square Number 1 and whose columns are the same as those of the 8×8 square above. This finally shows the uniqueness of the plane of order 8.

Although it has already been remarked that the first two cases above cannot be completed to full planes, it is a curious fact that a great many consistent lines may be added. These are listed here:

$$
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 0 & 4 & 7 & 5 & 3 & 2 & 6 \\
2 & 7 & 0 & 4 & 6 & 1 & 3 & 5 \\
7 & 5 & 3 & 6 & 2 & 4 & 1 & 0
\end{array}
\qquad
\begin{array}{cccccccc}
0 & 2 & 3 & 4 & 5 & 6 & 7 & 1 \\
3 & 6 & 0 & 5 & 1 & 4 & 2 & 7 \\
6 & 4 & 2 & 7 & 3 & 0 & 1 & 5 \\
7 & 1 & 5 & 2 & 6 & 3 & 0 & 4
\end{array}
$$

$$
\begin{array}{cccccccc}
0 & 3 & 4 & 5 & 6 & 7 & 1 & 2 \\
1 & 2 & 6 & 3 & 7 & 4 & 0 & 5 \\
4 & 1 & 0 & 7 & 2 & 6 & 5 & 3 \\
6 & 7 & 3 & 1 & 0 & 5 & 2 & 4
\end{array}
\qquad
\begin{array}{cccccccc}
0 & 4 & 5 & 6 & 7 & 1 & 2 & 3 \\
3 & 2 & 1 & 7 & 6 & 5 & 4 & 0 \\
4 & 0 & 3 & 2 & 1 & 7 & 6 & 5 \\
6 & 1 & 7 & 0 & 5 & 4 & 3 & 2
\end{array}
$$

$$
\begin{array}{cccccccc}
0 & 5 & 6 & 7 & 1 & 2 & 3 & 4 \\
2 & 1 & 3 & 5 & 7 & 0 & 4 & 6 \\
5 & 0 & 2 & 1 & 6 & 4 & 7 & 3 \\
7 & 3 & 1 & 0 & 4 & 6 & 2 & 5
\end{array}
\qquad
\begin{array}{cccccccc}
0 & 6 & 7 & 1 & 2 & 3 & 4 & 5 \\
1 & 4 & 3 & 0 & 6 & 2 & 5 & 7 \\
2 & 3 & 5 & 7 & 0 & 4 & 6 & 1 \\
5 & 1 & 6 & 4 & 3 & 7 & 2 & 0
\end{array}
$$

$$
\begin{array}{cccccccc}
0 & 7 & 1 & 2 & 3 & 4 & 5 & 6 \\
3 & 1 & 4 & 6 & 0 & 2 & 7 & 5 \\
4 & 5 & 7 & 3 & 6 & 0 & 2 & 1 \\
5 & 6 & 3 & 7 & 4 & 1 & 0 & 2
\end{array}
$$

Thus we have half of seven 8×8 squares. In projective terms we have a total of 45 lines with 9 points on a line which are consistent with themselves but which cannot be completed to the 73 lines of a projective plane.

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On the Location of Gauss Sums

We shall understand by a generalized Gauss sum of order k the sum

$$S_k = \sum_{m=0}^{p-1} \exp\,(2\pi i m^k/p), \quad (p = kf + 1, \text{ a prime}).$$

This sum can be thought of as the principal root z_0 of the reduced period equation of degree k for the so-called f-nomial periods, $z_i = k\eta_i + 1$, where, as usual,

$$\eta_i = \sum_{r=0}^{f-1} \exp\,(2\pi i g^{kr+i}/p) \quad (i = 0, 1, \cdots, k-1).$$

Since the remaining $k - 1$ roots of the period equation depend on the primitive root g, the singling out of one root z_0 as the principal root is justifiable.

For $k = 2$, it is well known that [1]

$$(1) \qquad\qquad S_2 = \begin{cases} \sqrt{p} & \text{if } p = 4n + 1 \\ i\sqrt{p} & \text{if } p = 4n - 1. \end{cases}$$

In general it is known that

$$- (k - 1)\sqrt{p} \leq S_k \leq (k - 1)\sqrt{p}.$$

For $k = 3$, and $4p = L^2 + 27M^2$, the three roots of the reduced trinomial cubic

$$F(z) = z^3 - 3pz - pL = 0 \quad (L \equiv 1 \pmod 3)$$

lie in the intervals

$$(-2\sqrt{p}, -\sqrt{p}), \quad (-\sqrt{p}, \sqrt{p}), \quad (\sqrt{p}, 2\sqrt{p}).$$

Kummer [2] noticed that the principal root z_0 lies in these intervals with frequencies approximately of 1 to 2 to 3 for primes less than 500 and conjectured that these ratios hold in general. Further calculations by Goldstine and von Neumann [3] for primes up to 10,000 exhibited ratios of the order of 2 to 3 to 4. They conclude that "these results would seem to indicate a significant departure from the conjectured densities and a trend toward randomness." On the other hand a case has been made by G. Beyer [4] that these are indeed the true ratios since $2 + 3 + 4 = 3^2$, and since these densities hold for composite as well as prime moduli. It therefore seemed of interest to continue this investigation on the SWAC at the Numerical Analysis Research of the University of California at Los Angeles. The results of Goldstine and von Neumann covering the first 611 primes were extended to the first 1000 primes of the form $6n + 1$, with the result that the ratios have now become of the order of 3 to 4 to 5, thus continuing the trend away from the conjecture and towards equal distribution noted by Goldstine and von Neumann. The actual figures will be given below.

Before doing this we would like to suggest another division of the primes of the form $6n + 1$ into three classes with respect to the absolute value of S_3. Since

$$F_3(-\sqrt{3p}) = F_3(\sqrt{3p}) = -pL,$$

there is always one root of $F_3 = 0$ which is greater than $\sqrt{3p}$ in absolute value, and we already know that there is a root of $F_3 = 0$ which is less than $\sqrt{p}$ in absolute value. Therefore the absolute value of S_3 lies in one of the three intervals

$$(0, \sqrt{p}), \quad (\sqrt{p}, \sqrt{3p}), \quad (\sqrt{3p}, 2\sqrt{p}).$$

We have actually counted the number of times that the principal root lies in these intervals using the tables of Goldstine and von Neumann and our own tabulation from the SWAC for primes above 10,000. We find that the number of primes in these three categories is approximately the same over the whole range covered. It is hoped that these figures will encourage some characterization of these classes of primes, which escapes the writer. The actual figures follow, where the primes are divided into 10 groups of 100 primes each.

TABLE 1. *Number of primes $p \equiv 1$ (mod 6) such that*

Group	$S_3 < -\sqrt{p}$	$-\sqrt{p} < S_3 < \sqrt{p}$	$S_3 > \sqrt{p}$	$|S_3| < \sqrt{p}$	$\sqrt{p} < |S_3| < \sqrt{3p}$	$|S_3| > \sqrt{3p}$
I	18	28	54	28	36	36
II	21	38	41	38	35	27
III	21	33	46	33	34	33
IV	29	32	39	32	32	36
V	28	29	43	29	35	36
VI	18	38	44	38	26	36
VII	34	26	40	26	31	43
VIII	24	32	44	32	41	27
IX	29	34	37	34	35	31
X	19	31	50	31	38	31
Total	241	321	438	321	343	336

A similar problem has been raised in the quartic case for the sum

$$S_4 = \sum_{m=0}^{p-1} \exp(2\pi i m^4/p)$$

by Hasse [5] in the closing pages of his book on the theory of numbers. The discussion of the problem, as given there, depends on much of what is in the book and an elementary treatment of it might be in order together with additional numerical evidence.

It was already known to Lebesgue [6] that S_4, as well as the companion sum

$$S_4' = \sum_{m=0}^{p-1} \exp(2\pi i r m^4/p),$$

where r is any quadratic, but not quartic, residue of p, satisfies the quartic equation

(2) $$[z^2 + (1 - (-1)^{(p-1)/4}2)p]^2 - 4p(z - a)^2 = 0,$$

where $p = a^2 + 4b^2$ and $p \equiv a \equiv 1$ (mod 4).

We can write (2) in factored form as follows

(3) $$[z^2 - 2\sqrt{p}z + (1 - (-1)^{(p-1)/4}2)p + 2\sqrt{p}a]$$
$$\times [z^2 + 2\sqrt{p}z + (1 - (-1)^{(p-1)/4}2)p - 2\sqrt{p}a] = 0.$$

Since by definition of S_2, S_4 and S_4' and by (1)

$$S_4 + S_4' = 2S_2 = 2\sqrt{p},$$

it follows that S_4 and S_4' are roots of the first factor of (3) and hence

(4) $$S_4, S_4' = p \pm \sqrt{2}\sqrt{(-1)^{(p-1)/4}p - \sqrt{p}a}.$$

We consider with Hasse the location of

(5) $$\Delta = \begin{cases} (S - S_4')/4 = \epsilon\sqrt{(p - \sqrt{p}a)/2} & \text{for } p = 8n + 1, \ \epsilon = \pm 1 \\ (S - S_4')/(4i) = \epsilon\sqrt{(p + \sqrt{p}a)/2} & \text{for } p = 8n + 5, \ \epsilon = \pm 1. \end{cases}$$

Besides the obvious ambiguity in the sign of ϵ, the sign of a is "known" only by reference to a table of quadratic partitions [7].

Hasse divides all primes of the form $4n + 1$ into four categories as follows:

$$p_1 = \begin{cases} 8n + 1 \\ 8n + 5 \end{cases} \text{for} \quad \begin{array}{l} \sqrt{2p}/2 < \Delta < \sqrt{p} \\ 0 < \Delta < \sqrt{2p}/2 \end{array} \quad \begin{array}{l} (\epsilon = +1, |a| \equiv -1 \pmod 4) \\ (\epsilon = +1, |a| \equiv -1 \pmod 4) \end{array}$$

$$p_3 = \begin{cases} 8n + 1 \\ 8n + 5 \end{cases} \text{for} \quad \begin{array}{l} -\sqrt{2p}/2 < \Delta < 0 \\ \sqrt{2p}/2 < \Delta < \sqrt{p} \end{array} \quad \begin{array}{l} (\epsilon = -1, |a| \equiv +1 \pmod 4) \\ (\epsilon = +1, |a| \equiv +1 \pmod 4) \end{array}$$

$$p_5 = \begin{cases} 8n + 1 \\ 8n + 5 \end{cases} \text{for} \quad \begin{array}{l} -\sqrt{p} < \Delta < -\sqrt{2p}/2 \\ -\sqrt{2p}/2 < \Delta < 0 \end{array} \quad \begin{array}{l} (\epsilon = -1, |a| \equiv -1 \pmod 4) \\ (\epsilon = -1, |a| \equiv -1 \pmod 4) \end{array}$$

$$p_7 = \begin{cases} 8n + 1 \\ 8n + 5 \end{cases} \text{for} \quad \begin{array}{l} 0 < \Delta < \sqrt{2p}/2 \\ -\sqrt{p} < \Delta < -\sqrt{2p}/2 \end{array} \quad \begin{array}{l} (\epsilon = +1, |a| \equiv +1 \pmod 4) \\ (\epsilon = -1, |a| \equiv +1 \pmod 4). \end{array}$$

These intervals become meaningful if one thinks of them as the projections on the x and y axis respectively of the first, third, fifth and seventh octants of the circle of radius $\sqrt{p}$. Hasse (there are two errors in Hasse's table, which he attributes to Kaluza. The primes 677 and 877 belong to class p_7 and not p_3, giving ratios $21:21:12:26$ instead of $21:23:12:24$) found that for primes less than 1000 the four types of primes appear with frequencies which are approximately $2:2:1:2$ for the primes of the form $8n + 1$ as well as for primes of the form $8n + 5$. He conjectured that there are infinitely many primes in each class and that the frequencies for primes of the forms $8n + 1$ are the same as for the primes of the form $8n + 5$.

Another, possibly more simple-minded, way of dividing these primes into four categories would be by grouping together all primes, irrespective of their form modulo 8 for which Δ lies in a given interval as follows:

(6)
$$\begin{array}{ll} p_0 \text{ for which } -\sqrt{p} < \Delta < -\sqrt{2p}/2 & (\epsilon = -1, |a| = -(-1)^{(p-1)/4} \pmod 4) \\ p_2 \text{ for which } -\sqrt{2p}/2 < \Delta < 0 & (\epsilon = -1, |a| = (-1)^{(p-1)/4} \pmod 4) \\ p_4 \text{ for which } 0 < \Delta < \sqrt{2p}/2 & (\epsilon = +1, |a| = (-1)^{(p-1)/4} \pmod 4) \\ p_6 \text{ for which } \sqrt{2p}/2 < \Delta < \sqrt{p} & (\epsilon = +1, |a| = -(-1)^{(p-1)/4} \pmod 4). \end{array}$$

If we now denote by $\pi_k(x)$ the number of primes in the class p_k not exceeding x, then obviously

$\pi_1(x) + \pi_5(x) = \alpha_1(x)$, the number of primes $< x$ for which $|a| = -1 \pmod 4$

$\pi_3(x) + \pi_7(x) = \alpha_3(x)$, the number of primes $< x$ for which $|a| = +1 \pmod 4$

$\pi_0(x) + \pi_6(x) = \alpha_0(x)$, the number of primes $< x$ for which $|a| = -(-1)^{(p-1)/4}$
$$\pmod 4$$

$\pi_2(x) + \pi_4(x) = \alpha_2(x)$, the number of primes $< x$ for which $|a| = (-1)^{(p-1)/4}$
$$\pmod 4.$$

It has been shown by Landau [8] that the Gaussian primes of the form $a + bi$, when divided into residue classes with respect to any modulus are asymptotically equally distributed. The Gaussian primes can be divided into two classes modulo 4, according as $|a| \equiv +1$, or $-1 \pmod 4$, $a \equiv 1 \pmod 4$, or according as $|a| \equiv \pm(-1)^{(p-1)/4} \pmod 4$. It therefore follows that

$$\lim_{x \to \infty} \alpha_3(x)/\alpha_1(x) = \lim_{a \to \infty} \alpha_2(x)/\alpha_0(x) = 1.$$

Since the ratios observed by Hasse at $x = 1000$ give

$$\alpha_1(1000)/\alpha_3(1000) = 3/4,$$

they cannot be the limiting ratios.

It should also be noted that by (6), α_0 gives the number of primes x, for which $|\Delta| > \sqrt{2p}/2$, while $\alpha_2(x)$ counts those for which $|\Delta| < \sqrt{2p}/2$; therefore $|\Delta|$ is in the limit equally likely to be in the two intervals $(0, \sqrt{2p}/2)$ and $(\sqrt{2p}/2, \sqrt{p})$. This could possibly be used to support the conjecture that the absolute value of S_3 is also equally divided between three intervals in the limit. However, the consideration of the Eisenstein primes of the form $a + b\omega$ modulo three does not seem to shed any light on the cubic problem any more than the consideration of the Gaussian primes modulo four helps us to find the limiting densities over the four intervals as taken by Hasse or as given in (6). We have therefore turned to the SWAC once more in order to calculate Δ for all primes of the form $4n + 1$ less than 10,000.

The time required by the SWAC for all 2549 primes for which Δ was calculated was about eight hours. Towards the end of the run, for primes in the neighborhood of 10,000 it took almost a minute per prime, but-considering that there were about 5000 sines or cosines to compute and add, not to speak of the calculation of the appropriate arguments, the time does not seem excessive. Also the accuracy of about 6 decimal places in cases which were computed by hand by (4) seemed very gratifying.

The results are as follows:

TABLE 2

x	$\pi_1(x)$	$\pi_3(x)$	$\pi_5(x)$	$\pi_7(x)$	$\pi_0(x)$	$\pi_2(x)$	$\pi_4(x)$	$\pi_6(x)$
1000	21	21	12	26	17	15	25	23
2000	42	34	27	44	23	26	47	41
3000	54	55	43	59	46	46	62	57
4000	77	68	49	75	54	55	84	76
5000	87	86	60	96	67	74	98	90
6000	105	99	71	108	78	84	112	109
7000	125	116	79	122	86	94	134	128
8000	145	126	93	135	101	106	151	141
9000	163	136	102	153	116	114	163	161
10000	179	151	113	166	125	131	176	177

These results do not change appreciably the picture obtained for primes less than one thousand. The frequency π_5 is still consistently low, but Hasse's ratios of

$2:2:1:2$ now look more like $3:3:2:3$, giving the ratio of $5/6$ for $\alpha_1(x)/\alpha_3(x)$ for $x = 10,000$, instead of $3/4$ at $x = 1000$, which is in line with the fact that this ratio tends to unity. The figures also seem to indicate that $\pi_0(x)$ and $\pi_2(x)$ are very close. The same can be said of $\pi_4(x)$ and $\pi_6(x)$ giving α_9/α_2 very close to one all along the line.

The next case, S_5, has not been studied in detail before, although the quintic period equation which it satisfies is known [9] (There is a misprint in equation (10) of [9]. A correct expression is (8) of the present paper.) in terms of the following quadratic partitions:

$$
\text{(7)} \qquad
\begin{aligned}
16p &= x^2 + 50u^2 + 50v^2 + 125w^2 \\
xw &= v^2 - u^2 - 4uv, \quad p \equiv x \equiv 1 \pmod 5
\end{aligned}
$$

and is as follows:

$$
\text{(8)} \quad F_5(z) = z^5 - 10pz^3 - 5pxz^2 + 5p[p - (x^2 - 125w^2)/4]z \\
+ p^2x - p[x^3 + 625(u^2 - v^2)w]/8 = 0.
$$

The roots of this equation lie between $-4\sqrt{p}$ and $4\sqrt{p}$, but when divided by $\sqrt{p}$ they no longer lie one each in any five fixed intervals. However, we can show by straightforward algebra that there always is a root z such that $|z| > \sqrt{5p}$, and that there are one, two, or three roots in the middle interval $|z| < \sqrt{p}$. Figures obtained for the five roots of (8) for primes less than 10,000 on the SWAC show that all of these cases do arise and that for approximately half the primes there are two roots in the middle interval, while for about a quarter of the primes there are one or three roots in the middle interval. The majority of cases with one root in the middle has one root in each interval, but there are exceptions. Nevertheless, the frequency of $|S_5|$ in these three intervals is surprisingly uniform, namely

TABLE 3

$p <$	$|S_5| < \sqrt{p}$	$\sqrt{p} < |S_5| < \sqrt{5p}$	$\sqrt{5p} < |S_5| < 4\sqrt{p}$	Total
1000	15	10	15	40
2000	30	19	24	73
3000	37	33	33	103
4000	46	43	45	134
5000	55	50	58	163
6000	67	60	67	194
7000	76	71	79	226
8000	84	76	86	246
9000	91	88	94	273
10000	108	97	101	306

These figures seem to be in line with the conjecture that $|S_5|$ is equally likely to lie in any one of the three intervals

$$(0, \sqrt{p}), \quad (\sqrt{p}, \sqrt{5p}), \quad (\sqrt{5p}, 4\sqrt{p}).$$

The distribution of S_5 itself over the corresponding 5 intervals gives at 10,000 the following numbers from left to right

$$33, \ 34, \ 108, \ 63, \ 68$$

with ratios of the order of $1:1:3:2:2$.

Another possible division into intervals is provided by the scaled projections of the fifth roots of unity on the x-axis. This gives $\pm \, (1 \pm \sqrt{5})$ for division points. For $p = 10,000$ the distribution from left to right over the five intervals is

$$12, \ 51, \ 122, \ 97, \ 24$$

or ratios of the order of $1:4:10:8:2$. Although the sum of these figures is temptingly 25, our experience with the cubic and quartic cases does not permit us to indulge in the hypothesis that these are indeed the limiting ratios. It is also hard to see how to interpret these figures as a generalization of Kummer's conjecture.

We have pursued a step further the conjecture about the equal distribution of $|S_k|$ for odd k in the intervals

$$(0, \sqrt{p}), \quad (\sqrt{p}, \sqrt{kp}), \quad (\sqrt{kp}, (k-1)\sqrt{p})$$

by calculating on the SWAC the values of S_7 for $p = 7n + 1 < 5000$ with the following results:

TABLE 4

$p <$	$|S_7| < \sqrt{p}$	$\sqrt{p} < |S_7| < \sqrt{7p}$	$\sqrt{7p} < |S_7| < 6\sqrt{p}$	Total
1000	6	11	10	27
2000	10	21	13	44
3000	16	32	21	69
4000	23	38	27	88
5000	33	43	32	108

These figures are not quite as convincing as those for the quintic, but since nothing is known about the distribution of the roots of the seventh degree period equation or about the form of the equation itself it is hoped that the actual tabulation of the roots as given by the SWAC will be of some use.

The actual values of these roots together with those of the quintic and the principal root of the cubic equation will be deposited in the UMT File of *MTAC*.

We wish to express our thanks to the Numerical Analysis Research Group of the University of California at Los Angeles for the use of the SWAC and in particular to Ruth Horgan and John Selfridge for checking out the various codes and operating the machine.

The code itself is diagrammed below. It presented no special difficulties and was put together from a standard cosine subroutine (a five term approximation of the 9th degree), which is used on the SWAC and fundamental number-theoretic subroutines, which will be described elsewhere.

Program for the calculation of $S_b = \sum_{m=1}^{p-1} \exp(2\pi i m^2)/p$

1. Input last prime p = kn + 1

2. p + n ⟶ p

3. Is p a prime?
   Yes   No

4. Find 1/p

5. Find $m^k \equiv \theta_m \pmod{p}$

6. Find $\cos 2\pi\theta_m/p$

7. Accumulate $\sum_{m=1}^{p-1} \cos 2\pi\theta_m/p$

8. m + 1 ⟶ m

9. Is m < p?
   Yes   No

10. Is $S^2 < p$?
    Yes   No

11. $-p \rightarrow p$

12. Does S_n check?
    Yes   No
              Stop

13. Output ±P, S_n
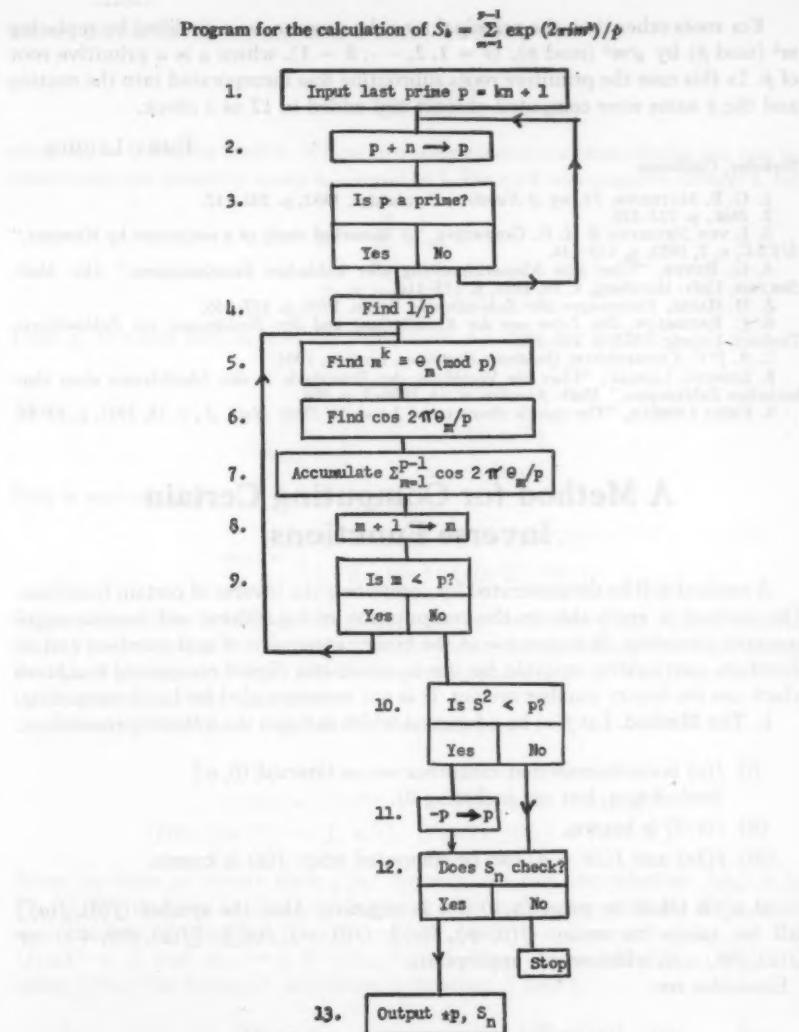
Box 3 uses the trial divisor test for primality.
Box 5 uses a standard reduction modulo p.
Box 6 uses the cosine subroutine.
Box 11 provides an indication on the size of the sum for output.
Box 12 consisted of either substituting the principal root into the equation, or adding all the roots to zero, in case all were computed. In a few cases, where the check failed, the calculation was repeated.

For roots other than the principal root the program was modified by replacing $m^k \pmod{p}$ by $\rho^\nu m^k \pmod{p}$, $(\nu = 1, 2, \cdots, k - 1)$, where ρ is a primitive root of p. In this case the primitive roots subroutine was incorporated into the routine and the k sums were computed abreast and added in 12 as a check.

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A Method for Computing Certain Inverse Functions

A method will be demonstrated for computing the inverse of certain functions. The method is applicable to the computation of logarithms and inverse trigonometric functions. It makes use of the binary expansion of real numbers and is, therefore, particularly suitable for use in automatic digital computing machines which use the binary number system. It is not recommended for hand computing.

1. The Method. Let $f(x)$ be a function which satisfies the following conditions,

(i) $f(x)$ is continuous and monotone on an interval $(0, a]$
  (including a, but not including 0),

(ii) $f(a/2)$ is known,

(iii) $f(2x)$ and $f(2x - a)$ can be computed when $f(x)$ is known.

$(0, a]$ is taken to mean $[a, 0)$ if a is negative. Also, the symbol $(f(0), f(a)]$ will be taken to mean $(f(0, +), f(a)]$ $(f(0, -), f(a)]$, $[f(a), f(0, +))$ or $[f(a), f(0, -))$, whichever is appropriate.

Examples are:

(a) $f(x) = 2^x$ $a = -1$
    $f(2x) = (f(x))^2$ $f(2x + 1) = 2(f(x))^2$

(b) $f(x) = \cos x$ $a = \pi$
    $f(2x) = 2(f(x))^2 - 1$ $f(2x - \pi) = 1 - 2(f(x))^2$

Let $y \,\epsilon\, (f(0), f(a)]$, and let it be required to compute $f^{-1}(y)$, that is, to find x such that $f(x) = y$. The existence and uniqueness of such an x in $(0, a]$ are guaranteed by condition (i). Let $w = x/a$. Then w is in the interval $(0, 1]$. It

follows that w has an expansion as a binary number,

$$w = \sum_{n=1}^{\infty} u_n \cdot 2^{-n}$$

where $u_n = 0$ or 1 for each n. Where two such expansions exist, choose the one in which there are infinitely many u_n's equal to 1. For each non-negative integer k, let

$$w_k = \sum_{n=1}^{\infty} u_{n+k} \cdot 2^{-n},$$

$$x_k = w_k a.$$

Then $x_0 = x$ and each x_k is in the interval $(0, a]$. Also, for each k,

$$x_k = \tfrac{1}{2} u_{k+1} a + \tfrac{1}{2} x_{k+1}.$$

Thus,

$$u_{k+1} = \begin{cases} 0 & \text{if} \quad x_k \in (0, a/2] \\ 1 & \text{if} \quad x_k \in (a/2, a]. \end{cases}$$

This is equivalent to

$$u_{k+1} = \begin{cases} 0 & \text{if} \quad f(x_k) \in (f(0), f(a/2)] \\ 1 & \text{if} \quad f(x_k) \in (f(a/2), f(a)]. \end{cases}$$

By condition (iii), it follows that $f(x_{k+1})$ can be computed when $f(x_k)$ is known. Since $f(x_0) = y$ is given, an inductive procedure is established for determining the u_k's. These determine w, and the required number x is wa. The procedure will be illustrated in three examples:

Example 1. Compute $\log_2 (.6)$. In this example,

$$f(x) = 2^x \qquad\qquad a = -1$$
$$f(2x) = (f(x))^2 \qquad f(2x + 1) = 2(f(x))^2$$
$$(f(0), f(a/2)) = (1, \sqrt{.5}) \quad (f(a/2), f(a)) = (\sqrt{.5}, .5].$$

Since we have to square each $f(x_k)$ anyway, we can test whether $f(x_k)$ is in $(0, f(a/2)]$ or $(f(a/2), f(a)]$ by squaring and comparing with .5. Thus, each step consists of squaring $f(x_k)$, comparing with .5 and recording $u_{k+1} = 1$ if $(f(x_k))^2 < .5$, and $u_{k+1} = 0$ if $(f(x_k))^2 \geq .5$. Then we record $f(x_{k+1})$ which is either $(f(x_k))^2$ or $2(f(x_k))^2$, whichever is between .5 and 1.

k	$f(x_k)$	$(f(x_k))^2$	u_{k+1}
0	.6	.36	1
1	.72	.5184	0
2	.5184	.26873 856	1
3	.53747 712	.28888 16545 23494 4	1
4	.57776 33090 46988	$\cdots$	

Thus $w = .1011 \cdots$ (binary notation) and $\log_2 .6 = aw = -.1011 \cdots$ (binary)
$= -(11/16 + \epsilon) \quad 0 < \epsilon < 1/16.$

Example 2. Compute arc cos 0.5. In this example,

$$f(x) = \cos x \qquad\qquad a = \pi$$
$$f(2x) = 2(f(x))^2 - 1 \qquad f(2x - \pi) = 1 - 2(f(x))^2$$
$$(f(0), f(a/2)] = (1, 0] \qquad (f(a/2), f(a)] = (0, -1]$$

At each step we record $u_{k+1} = 0$ if $f(x_k) \geq 0$, $u_{k+1} = 1$ if $f(x_k) < 0$. Then $f(x_{k+1}) = \pm (2(f(x_k))^2 - 1)$, the sign $\pm$ is that of $f(x_k)$.

k	$f(x_k)$	u_{k+1}	$2(f(x_k))^2 - 1$
0	.5	0	−.5
1	−.5	1	−.5
2	.5	0	−.5
3	−.5	1	−.5

$$w = .0101 \cdots \text{ (binary)},$$
$$w = 1/3$$
$$\text{arc cos } 0.5 = \pi/3.$$

Example 3. Compute arc cos 0.2.

k	$f(x_k)$	u_{k+1}	$2(f(x_k))^2 - 1$
0	.2	0	−.92
1	−.92	1	.6928
2	−.6928	1	−.04005 632
3	.04005 632	0	$\cdots$

$$w = .0110 \cdots \text{ (binary)}$$
$$w = 3/8 + \epsilon, 0 < \epsilon < 1/16$$
$$\text{arc cos } 0.2 = w\pi = 3\pi/8 + \epsilon\pi, \quad 0 < \epsilon\pi < 0.1964.$$

2. Error Analysis. Condition (iii) is seldom realized in practice, for required quantities are usually "computable" only approximately. Even so simple a process as squaring cannot be repeated indefinitely without eventually "rounding off." A more practical condition than (iii) is

(iiia) there exist small positive numbers δ and ϵ, such that when $f(x)$ is given, $(x \epsilon (0, a])$ numbers can be computed which differ from $f(2x)$ and $f(2x - a)$, respectively, by less than ϵ, and such that for any pair x, $x' \epsilon (0, a]$, $|f(x) - f(x')| < \epsilon$ implies $|x - x'| < \delta$.

Hereafter ϵ will designate the *maximum roundoff error*. Suppose that the calculation of x proceeds as indicated above, except that at each stage the true value of $f(x_n)$ is replaced by an approximation y_n, computed in accordance with (iiia). Then for each n,

$$|y_n - f(2f^{-1}(y_{n-1}) - au_n)| < \epsilon$$

and this implies

$$|f^{-1}(y_n) - 2f^{-1}(y_{n-1}) + au_n| < \delta.$$

Now, let

$$z_0 = x$$

and for each positive integer N, let

$$z_N = a \sum_{n=1}^{N} u_n \cdot 2^{-n} + 2^{-N} f^{-1}(y_N).$$

That is, z_N consists of the first N binary digits of w plus an error term. Then, for each n,

$$z_n - z_{n-1} = 2^{-n}(f^{-1}(y_n) - 2f^{-1}(y_{n-1}) + au_n)$$
$$|z_n - z_{n-1}| < 2^{-n}\delta.$$

Thus for every N

$$|z_N - x| = |z_N - z_0|$$
$$= \left| \sum_{n=1}^{N} (z_n - z_{n-1}) \right| < \sum_{n=1}^{N} 2^{-n}\delta < \delta.$$

Thus each z_n differs from the required x by less than δ. If the process could be carried on indefinitely, therefore, roundoff errors would introduce a total error no greater than δ. Since δ is a function of the computation facilities available and independent of the number of steps executed, it will be designated as the *maximum residual error*. It is interesting to note that the maximum residual error is numerically equal to the maximum error that could result from rounding the input number, y.

If the process is terminated after N steps, the result is an underestimate and should, therefore, be "rounded up" by adding $2^{-N-1}a$. The computed value for x is, then,

$$a\left(\left(\sum_{n=1}^{N} u_n \cdot 2^{-n} \right) + 2^{-N-1} \right).$$

This differs from z_N by

$$2^{-N}(f^{-1}(y_N) - a/2)$$

which is not greater in absolute value than $2^{-N-1}a$. Therefore, $2^{-N-1}|a|$ is designated the *maximum termination error*. The total error is obviously less than the sum of the maximum residual error and the maximum termination error.

In Example 1 above ($\log_2 y$), the maximum residual error is $1/2(2^\epsilon - 1)$ where ϵ is the maximum roundoff error. It is less than

$$1/2 \frac{\epsilon \log_e 2}{1 - \epsilon \log_e 2}.$$

The maximum termination error is 2^{-N-1}.

In Examples 2 and 3 above, arc cos y, the maximum residual error is arc cos $(1 - \epsilon)$, which is approximately $\sqrt{2\epsilon}$, and the maximum termination error is $\pi/2^{N+1}$.

3. Programming for Machine Computation. A typical machine program for the computation of $f^{-1}(x)$ by the method of section 1 would be as follows:

(It is assumed that $f(x)$ is an *increasing* function of x. If it is a *decreasing* function, then the relation $>$ should be replaced by $<$ in Step 1.) The symbol C_m means "contents of cell m."

Cell Number	Initial Contents	Representing
1	y	y_n
2	$1/2$	2^{-n-1}
3	0	$\sum_{i=1}^{n} u_i \cdot 2^{-i}$
4	2^{-N}	2^{-N}
5	$f^{-1}(a/2)$	$f^{-1}(a/2)$

Program:

    Step 1. If $C_1 > C_5$, proceed to Step 3. If not, proceed to Step 2.
    Step 2. Compute $f(2f^{-1}(C_1))$ and store in cell 1. Proceed to Step 5.
    Step 3. Compute $f(2f^{-1}(C_1) - a)$ and store in cell 1. Proceed to Step 4.
    Step 4. Store $C_2 + C_3$ in cell 3. Proceed to Step 5.
    Step 5. Store $1/2C_2$ in cell 2. Proceed to Step 6.
    Step 6. If $C_2 < C_4$, proceed to Step 7. If not, proceed to Step 1.
    Step 7. Store $C_2 + C_3$ in cell 3. Store $a \cdot C_3$ in cell 3.
        The computed value of $f^{-1}(y)$ is C_3.

Of course, the program outlined above can be improved when special properties of $f(x)$ are known. If, for example, $f(x)$ is monotone in the interval $(0, 2a]$ then comparing of $f(x)$ and $f(a/2)$ is equivalent to comparing $f(2x)$ and $f(a)$. (This condition is satisfied by 2^x, but not by $\cos x$.) If in addition, $f(x - a)$ can be computed when $f(x)$ is known, the program outlined above can be modified by placing $f^{-1}(a)$ rather than $f^{-1}(a/2)$ in cell 5, and altering steps 1–3 to read:

    Step 1. Compute $f(2f^{-1}(C_1))$ and store in cell 1. ·
    Step 2. If $C_1 > C_5$, proceed to Step 3. If not, proceed to Step 5.
    Step 3. Compute $f(f^{-1}(C_1) - a)$ and store in cell 1. Proceed to Step 4.

The modified program has several advantages. For one thing, Step 3 of the modified program is likely to require fewer orders than Step 3 of the original program. In the case of the function $f(x) = 2^x$, for example, Step 3 of the modified program is multiplication by 2, whereas in the original program, it is squaring and multiplying by 2. Another advantage in the case $f(x) = 2^x$ is that $f(a) = 1/2$ is more easily and accurately evaluated than $f(a/2) = \sqrt{1/2}$.

It may be possible to take advantage of special features of the particular

machine to be used, by combining certain steps. A program for $\log_2 x$ has been developed for the CRC-102A Computer, which has only four commands in the repeating loop, and only three additional cells referred to in the repeating loop. The repetitive part of the program is shown below, coded in octal, with explanatory notes at the right.

Cell Number		(Initial)	Contents		
2000	25	2007	2007	2007	Square y
2001	27	2006	2005	2006	Shift logical
2002	31	2006	2007	2006	Scale factor
2003	37	2006	2003	2000	Test overflow
2004	34	3000	2100	—*	Exit
2005	00	0000	0000	0001	Shift control
2006	37	7777	7777	7776	See text
2007				y	y_a

* Address of next command in main program.

The number in cell 2006 is

$$2^{a-35}\left(1 - \tfrac{1}{2}\sum_{i=1}^{a} u_i \cdot 2^{-i}\right) - 1.$$

4. Other Inverse Trigonometric Functions. Although the method given above is applicable directly to the computation of other inverse trigonometric functions, the functions $f(2f^{-1}(y) - u_n a)$ are in general more complicated than $\pm (2y^2 - 1)$. It is advantageous, therefore, to evaluate other inverse trigonometric functions by converting them to inverse cosines, as follows:

$$\text{arc sin } y = \pi/2 - \text{arc cos } y$$

$$\text{arc tan } y = \pm 1/2 \text{ arc cos} \left(\frac{1 - y^2}{1 + y^2}\right)$$

(sign is that of y)

$$\text{arc cot } y = \pi/2 \pm 1/2 \text{ arc cos} \left(\frac{1 - y^2}{1 + y^2}\right)$$

(sign opposite that of y)

$$\text{arc sec } y = \text{arc cos } (1/y)$$

$$\text{arc csc } y = \frac{\pi}{2} \text{ arc cos } (1/y).$$

The application to arc sin y provides an interesting algorithm for computing arc sin y, which may be useful even in hand computing. The algorithm is as follows:

1. Iterate the function $2y^2 - 1$, starting with the number y whose arc sin is required.
2. Record the *signs* of the iterates in order.
3. Accumulate the signs; that is, record the "partial products" of the signs in order.
4. Write descending powers of 2 between the signs accumulated.
5. Multiply the series obtained by $\pi/2$.

Example: Compute arc sin $\sqrt{.75}$

1. $\sqrt{.75}$,   .5, $-.5$, $-.5$, $-.5$ $\cdots$
2. $+$ $+$ $-$ $-$ $-$ $\cdots$
3. $+$ $+$ $-$ $+$ $-$ $\cdots$
4. $+1/2$ $+1/4$ $-1/8$ $+1/16$ $-1/32$ $\cdots$ $= 2/3$
5. $\pi/2 \cdot (2/3) = \pi/3$
    arc sin $\sqrt{.75} = \pi/3$.

5. Comparison with Other Methods. The usefulness of the method described above as compared with other methods depends, of course, on the function to be evaluated and on the features of the machine to be used.

The fact that each iteration yields exactly one binary bit may be an advantage or a disadvantage; a method where error decreases faster than 2^{-n} will converge with fewer iterations than this one. On the other hand, one iteration of this method may consist of fewer commands than an iteration of another method. The logarithm program for the CRC-102A described above has 4 commands per iteration as compared with 14 commands per iteration in another program for logarithm on the same machine. The program for arcsin by the method described above has 9 commands per iteration as compared to 22 for another arcsin program. The fact that each iteration yields exactly one binary bit also simplifies error analysis, for the number of digits of accuracy is exactly one less than the number of digits computed.

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A Method for Solving Algebraic Equations using an Automatic Computer

Introduction. Many methods have been developed for solving algebraic equations and several of these have been used with automatic computers [1, 2]. Those methods which are most suitable for use with automatic computers are ones which apply to a wide class of equations and which are relatively rapid when the degree of the equation is large. The method described here has been constructed with these considerations in mind and has been programmed for the ILLIAC computer at the University of Illinois.

A process is to be constructed to find n solutions to the general algebraic equations of nth degree

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0 \tag{1}$$

where the coefficients $a_0, a_1, \cdots, a_n$ are complex numbers and $a_0 \neq 0$. Each root is found by an iterative procedure. Successive iterations toward a particular root are obtained by finding the nearer root of a quadratic whose curve passes through the last three points. The quadratic will in general have complex coefficients and complex roots. This solution is accomplished by a variation of the standard quadratic formula. Although the method derived here is rather complicated, no evaluation of derivatives of $f(x)$ and only one evaluation of the polynomial $f(x)$ is required per iteration. If the degree of the equation is large a greater amount of time is spent evaluating the function than is spent in the remainder of the process. Thus, the time spent per iteration is less with this process than with iterative schemes which require the calculation of derivatives, whenever the degree of the equation is large.

The Lagrange interpolation formula will yield a quadratic

$$L_i(f(x)) = b_0 x^2 + b_1 x + b_2 \tag{2}$$

whose curve passes through the last three points $(x_i, f(x_i))$, $(x_{i-1}, f(x_{i-1}))$, $(x_{i-2}, f(x_{i-2}))$ where the coefficients b_0, b_1, b_2 satisfy the system of equations

$$\begin{aligned} b_0 x_i^2 + b_1 x_i + b_2 &= f(x_i), \\ b_0 x_{i-1}^2 + b_1 x_{i-1} + b_2 &= f(x_{i-1}), \\ b_0 x_{i-2}^2 + b_1 x_{i-2} + b_2 &= f(x_{i-2}). \end{aligned} \tag{3}$$

A somewhat more convenient representation for this quadratic is obtained by introducing the new quantities $h = x - x_i$, $h_i = x_i - x_{i-1}$, $h_{i-1} = x_{i-1} - x_{i-2}$, $\lambda = h/h_i$, $\lambda_i = h_i/h_{i-1}$, and $\delta_i = 1 + \lambda_i$. The Lagrange interpolation formula may now be written as the following quadratic in λ.

$$L_i(f(x)) = \lambda^2 \delta_i^{-1} [f(x_{i-2})\lambda_i^2 - f(x_{i-1})\lambda_i \delta_i + f(x_i)\lambda_i] \tag{4}$$
$$+ \lambda \delta_i^{-1}[f(x_{i-2})\lambda_i^2 - f(x_{i-1})\delta_i^2 + f(x_i)(\lambda_i + \delta_i)] + f(x_i).$$

A single iterative step is obtained by letting x_{i+1} be a value of x which makes $L_i(f(x))$ vanish. We may solve the quadratic equation in λ obtained by setting expression (4) equal to zero. We then obtain $\lambda = \lambda_{i+1} = (x_{i+1} - x_i)/(x_i - x_{i-1})$ by using the inverse of the standard quadratic formula

$$\lambda_{i+1} = \frac{-2f(x_i)\delta_i}{g_i \pm \sqrt{g_i^2 - 4f(x_i)\delta_i \lambda_i [f(x_{i-2})\lambda_i - f(x_{i-1})\delta_i + f(x_i)]}} \tag{5}$$

where $g_i = f(x_{i-2})\lambda_i^2 - f(x_{i-1})\delta_i^2 + f(x_i)(\lambda_i + \delta_i)$. From λ_{i+1} we may obtain $h_{i+1} = \lambda_{i+1}h_i$ and $x_{i+1} = x_i + h_{i+1}$. This x_{i+1} represents a zero of the quadratic described above. The sign in the denominator of (5) is always taken so as to make the denominator have the greater magnitude. This makes λ_{i+1} and h_{i+1} each be

that one of the two possible choices having the smaller magnitude so that x_{i+1} is the root which is closer to x_i.

A convenient starting method for this process uses artificial starting values at $x_0 = -1$, $x_1 = 1$, and $x_2 = 0$:

$$\begin{aligned}
&a_n - a_{n-1} + a_{n-2} \quad \text{is used for } f(x_0),\\
(6) \quad &a_n + a_{n-1} + a_{n-2} \quad \text{is used for } f(x_1),\\
&a_n \qquad\qquad\qquad\quad \text{is used for } f(x_2).
\end{aligned}$$

thus making $\lambda_2 = -\tfrac{1}{2}$ and $h_2 = -1$. This choice of starting values makes

$$L_2(f(x)) = a_n + a_{n-1}x + a_{n-2}x^2$$

which approximates to $f(x)$ in the neighborhood of $x = 0$. The advantage of this starting process is that it requires no special evaluations of the polynomial $f(x)$ and is therefore rapid.

Convergence of the Process. A final value of the root x_i is taken when $|x_i - x_{i-1}|/|x_i|$ becomes less than some preassigned number. Such a convergence criterion is consistent with the use of floating point arithmetic in the calculation. As a result of this criterion we see that convergence occurs if $x_{i-1} = x_i$. This means that before convergence no two consecutive iterative results will be equal. Furthermore, if $x_i = x_{i-2}$ we have $\delta_i = \dfrac{x_i - x_{i-2}}{x_{i-1} - x_{i-2}} = 0$ so by (5) $\lambda_{i+1} = 0$ and $x_{i+1} = x_i$ also giving convergence unless $x_i = 0$. Thus in normal operation of the process we see that x_i, x_{i-1}, and x_{i-2} are distinct.

As each root is found it may be divided into the polynomial $f(x)$ thus reducing the degree of the polynomial by one. The algorithm for this reduction is the commonly used one

$$(7) \qquad\qquad a_i' = ra_{i-1} + a_i, \quad (i = 0, 1, 2, \cdots)$$

where a_i' is the new coefficient to replace a_i and r is the root which has just been found. We make $a_{-1} = 0$. Errors introduced by this process will be reduced if the roots are eliminated in order of increasing magnitude. By always starting at the point $x = 0$ one will tend to find roots in roughly this order.

No general proof of convergence in the large has been obtained for this process, but convergence can be shown to occur whenever the process leads one sufficiently close to a single or double root.

In order to facilitate the study of convergence let us assume that $x_{i+1} = 0$. This loses no generality since a simple shift of origin is always possible within the system. At point x_{i+1} we also have $L_i(x_{i+1}) = 0$ so that

$$(8) \qquad\qquad L_i(x_{i+1}) = b_0x^2_{i+1} + b_1x_{i+1} + b_2 = b_2 = 0.$$

Now each of the functions appearing on the right hand sides of equations (3) may be expanded about $x_{i+1} = 0$ as

$$(9) \qquad\qquad f(x_{i-m}) = \sum_{k=0}^{n} x^k_{i-m} f^{(k)}(0)/k!$$

and b_2 may be written as

$$(10) \qquad b_2 = \sum_{k=0}^{\infty} b_{2k} f^{(k)}(0)/k! = 0$$

where b_{2k} is obtained by solving the system of equations

$$(11) \qquad \begin{aligned} b_{0k}x_i^2 + b_{1k}x_i + b_{2k} &= x_i^k \\ b_{0k}x^2_{i-1} + b_{1k}x_{i-1} + b_{2k} &= x^k_{i-1} \\ b_{0k}x^2_{i-2} + b_{1k}x_{i-2} + b_{2k} &= x^k_{i-2}. \end{aligned}$$

This system of equations may be solved by elimination provided x_i, x_{i-1}, and x_{i-2} are distinct and we obtain $b_{20} = 1$, $b_{21} = b_{22} = 0$ for the first three solutions. When $k \geq 3$ the solution becomes somewhat more difficult but may be carried out by eliminating b_{1k} between the first two equations to give

$$b_{0k}x_i x_{i-1} - b_{2k} = x_i x_{i-1} \sum_{p=0}^{k-2} x_i^p x_{i-1}^{k-2-p}.$$

A similar elimination may be made between another pair and the result combined with the above equations to give

$$(12) \qquad b_{2k} = x_i x_{i-1} x_{i-2} \sum_{p+q+s=k-3} x_i^p x_{i-1}^q x_{i-2}^s.$$

The sum is to be taken over all non-negative integral p, q, and s satisfying $p + q + s = k - 3$. All of these results may be inserted in (8) and (10) giving

$$(13) \qquad f(0) = -x_i x_{i-1} x_{i-2} \left(\tfrac{1}{6}f'''(0) + \sum_{k=4}^{\infty} \sum_{p+q+s=k-3} x_i^p x_{i-1}^q x_{i-2}^s f^{(k)}(0)/k! \right).$$

Up to this point no approximations have been made and no limits have been taken. Equation (13) expresses the same relationship contained in (5). We now assume that the points x_i, x_{i-1}, x_{i-2} lie in the neighborhood of a root r. Thus if we let $\epsilon_{i+1} = x_{i+1} - r$, $\epsilon_i = x_i - r$, $\epsilon_{i-1} = x_{i-1} - r$ and $\epsilon_{i-2} = x_{i-2} - r$ the magnitudes of the last three quantities are all assumed to be less than some upper bound ϵ_m

$$(14) \qquad |\epsilon_i| < \epsilon_m, \quad |\epsilon_{i-1}| < \epsilon_m, \quad |\epsilon_{i-2}| < \epsilon_m.$$

We also now make the tentative assumption

$$(15) \qquad |\epsilon_{i+1}| < \epsilon_m,$$

and we shall seek to justify this assumption later. If (14) and (15) are inserted in (13) and the functions are expanded about r we obtain

$$(16) \quad \epsilon_{i+1}f'(r) + \epsilon_{i+1}^2 \tfrac{1}{2}f''(r) + \epsilon_{i+1}^3 \tfrac{1}{6}f'''(r)$$
$$= -(\epsilon_i - \epsilon_{i+1})(\epsilon_{i-1} - \epsilon_{i+1})(\epsilon_{i-2} - \epsilon_{i+1})\tfrac{1}{6}f'''(r) + 0(\epsilon_m^4).$$

If r is a single root we see that $\epsilon_{i+1} = 0(\epsilon_m^3)$ so that we may write

$$(17) \qquad \epsilon_{i+1} = -\epsilon_i \epsilon_{i-1} \epsilon_{i-2} \frac{f'''(r)}{6f'(r)} + 0(\epsilon_m^4).$$

A solution ϵ_{i+1} to this equation does exist if ϵ_m is sufficiently small and will satisfy (15). This solution will also satisfy $L_i(f(x_{i+1})) = 0$, and hence we are justified in assumption (15) and hence (17) for at least one of the two x_{i+1} for which $L_i(f(x_{i+1})) = 0$ holds. We now wish to show that (17) holds for that x_{i+1} which is actually chosen by the process described in connection with equation (5). If both x_{i+1} satisfy (17), the proof need not be given. If, however, one does and one does not, we must make some further analysis. It was pointed out that the process chooses the point x_{i+1} which is nearer to x_i. The point given by equation (17) must satisfy

$$(18) \qquad |x_{i+1} - x_i| = |\epsilon_{i+1} - \epsilon_i| < 2\epsilon_m.$$

This must therefore also hold for the x_{i+1} which is chosen by the process in (5) and hence $|\epsilon_{i+1}| < 3\epsilon_m$ for this case. But $|\epsilon_{i+1}| < 3\epsilon_m$ is adequate to give (17) for sufficiently small ϵ_m and we may therefore assume (17) for the ϵ_{i+1} obtained in the process.

A general limiting formula for the ϵ_j in the neighborhood of a root may be obtained from equation (17). If logarithms are taken on both sides we obtain

$$(19) \qquad \log \epsilon_{i+1} = \log \epsilon_i + \log \epsilon_{i-1} + \log \epsilon_{i-2} + \log \left(- \frac{f'''(r)}{6f'(r)} \right) + 0(\epsilon_m).$$

Neglecting the terms $0(\epsilon_m)$ we may solve (19) as a difference equation using standard techniques and obtain

$$(20) \qquad \log \epsilon_j = c_1 m_1{}^j + c_2 m_2{}^j + c_3 m_3{}^j - \tfrac{1}{2} \log \left(- \frac{f'''(r)}{6f'(r)} \right).$$

Where the constants c_1, c_2, and c_3 are determined by the starting values and the three orders of convergence m_1, m_2, and m_3 are roots of the characteristic equation

$$(21) \qquad m^3 = m^2 + m + 1.$$

The roots are

$$m_1 = 1.84,$$
$$m_2, m_3 = -.420 \pm .606i.$$

Since the last two roots have magnitude less than 1 their effect will die out and the order of the process is given by m_1. After these approximations become valid we have from (20)

$$(22) \qquad \epsilon_{j+1} = K\epsilon_j{}^{1.84}$$

where
$$K = \left(- \frac{f'''(r)}{6f'(r)} \right)^{-\frac{1}{2}}.$$

In the case of a double root a similar argument exists. Equation (17) is then replaced by

$$(23) \qquad \epsilon_{i+1}^2 = -\epsilon_i \epsilon_{i-1} \epsilon_{i-2} \frac{f'''(r)}{3f''(r)} + 0(\epsilon_m^4)$$

and the characteristic equation becomes

$$(24) \qquad 2m^3 = m^2 + m + 1.$$

It has roots

$$m_1 = 1.23,$$
$$m_2, m_3 = -.367 \pm .520i,$$

and again the order of convergence is given by m_1. We therefore have in the limit

$$(25) \qquad \epsilon_{j+1} = K\epsilon_j^{1.23}$$

with

$$K = \left(- \frac{f'''(r)}{3f''(r)} \right)^{-\frac{1}{2}}.$$

Convergence of the Generalized Process. One might imagine a generalized process in which an α degree Lagrange interpolation polynomial $L_i(x, \alpha)$ is used rather than the quadratic of equation (2). This presumes that some new method for obtaining the nearest root of this polynomial is to be used. Since the direct method corresponding to equation (5) would no longer be practical, one would probably use some iterative method for solution of the equation $L_i(x_{i+1}, \alpha) = 0$.

We now wish to investigate the convergence rate for such a process.

A general set of equations corresponding to equations (11) may be formed. They are

$$(26) \qquad \sum_{j=0}^{\alpha} b_{jk} x_{i-s}^{\alpha-j} = x_{i-s}^k, \qquad s = 0, 1, \cdots, \alpha.$$

When $k > \alpha$ the quantity $b_{\alpha k}$ may be obtained by elimination. (This direct method for obtaining $b_{\alpha k}$ was pointed out to the author by Mr. W. Scott Bartky.) Let us eliminate $b_{\alpha-1,k}$ between the first equation and each succeeding equation giving

$$(27) \quad \sum_{j=0}^{\alpha-2} b_{jk} x_i x_{i-s} (x_i^{\alpha-j-1} - x_{i-s}^{\alpha-j-1}) - b_{\alpha k}(x_i - x_{i-s}) = x x_{i-s} (x_i^{k-1} - x_{i-s}^{k-1}),$$

$$s = 1, 2, \cdots, \alpha.$$

Each equation may be divided by $(x_i - x_{i-s})$ giving

$$(28) \quad \sum_{j=0}^{a-2} b_{jk} x_i x_{i-s} \sum_{l=0}^{a-j-2} x_i^l x_{i-s}^{a-2-j-l} - b_{ak} = x_i x_{i-s} \sum_{l=0}^{k-2} x_i^l x_i^{k-2-l}, \quad s = 1, 2, \cdots, \alpha.$$

We next eliminate $b_{a-2,k} \cdots, b_{0k}$ in a similar manner until the result

$$(29) \quad b_{ak} = (-1)^a x_i x_{i-1} \cdots x_{i-a} \sum x_i^{p_0} x_{i-1}^{p_1} \cdots x_{1-a}^{p_a},$$
$$p_0 + p_1 + \cdots + p_a = k - (\alpha + 1)$$

is obtained. In this expression the summation is made over all terms for which the exponents $p_0, p_1, \cdots, p_a$ are non-negative integers and $\sum_{j=0}^{\alpha} p_j = k - (\alpha + 1)$. We also see directly from (26) that $b_{ak} = 0$ if $k \leq \alpha$, except that $b_{a0} = 1$.

We may therefore obtain a generalization of equation (17)

$$(30) \quad \epsilon_{i+1} = (-1)^{\alpha+1} \epsilon_i \epsilon_{i-1} \cdots \epsilon_{i-a} \frac{f^{(\alpha+1)}(r)}{(\alpha + 1)! f'(r)} + 0(\epsilon_m^{\alpha+2})$$

and obtain for the characteristic equation corresponding to (21)

$$(31) \quad m^{\alpha+1} = m^\alpha + m^{\alpha-1} + \cdots + 1$$

This equation has one root m_1 which lies between 1 and 2 on the real axis and which approaches 2 with increasing α. The remaining roots lie within the unit circle and therefore represent perturbations which die out. The order of convergence of the process to single roots is therefore given by m_1. Since this can never reach 2 we conclude that there is little to be gained in speed of convergence by letting α exceed 2.

One should not ignore the possibility of letting $\alpha = 1$. In this case the formula corresponding to (5) is greatly simplified since a linear equation rather than a quadratic now must be solved. This choice, however, suffers from a disadvantage if all the coefficients of the original equation are real. If one starts from a real point x_0 then all successive iterative results x_i will also be real and hence only real roots will be found.

Tests of the Method. The process with $\alpha = 2$ as outlined in the preceding sections was altered slightly in practice. Whenever the new value of the function $f(x_{i+1})$ is calculated the quantity $|f(x_{i+1})|/|f(x_i)|$ is formed. If this latter quantity exceeds 10 the quantity λ_{i+1} is halved and $h_{i+1} x_{i+1}$, and $f(x_{i+1})$ are recomputed accordingly. With this revision the process has produced convergence in all the cases tested.

Another alteration was made to handle the case in which the denominator of (5) is zero. This occurs whenever $f(x_i) = f(x_{i-1}) = f(x_{i-2})$ and in such cases the arbitrary value $\lambda_{i+1} = 1$ is chosen since (5) may no longer be used.

The process ($\alpha = 2$) was tested for equations of varying degree. Fourteen equations were solved starting with degree 10 and progressing in steps of ten through degree 140. Each equation was formed by choosing random points as

roots within the square having vertices $\pm 1 \pm i$. Polynomials were then formed from these roots. The solutions to these polynomials were then compared with the original random numbers which were used to generate the polynomial. Results are summarized in the table.

Degree of Equation	Accuracy of Least Accurate Root	Accuracy of Last Root to be Found	Time Taken by ILLIAC for Complete Solution in Minutes
10	10^{-7}	10^{-9}	1
20	10^{-8}	10^{-8}	2
30	10^{-5}	10^{-8}	5
40	10^{-4}	10^{-9}	6
50	10^{-4}	10^{-6}	10
60	10^{-5}	10^{-7}	12
70	10^{-4}	10^{-5}	17
80	10^{-4}	10^{-5}	20
90	10^{-1}	10^{-5}	20
100	—	10^{-6}	33
110	—	—	42
120	—	—	43
130	—	10^{-8}	48
140	—	—	60

Dashes in the table indicate that some roots were too inaccurate to be identified. In all equations some roots appeared which were correct to 10^{-8} or better. The solutions to the 100th degree equation, which had some unidentifiable roots, were used to generate a polynomial. All coefficients of this polynomial agreed with the coefficients of the original polynomial to at least 6 decimal places. This result indicates that the obtaining of accurate values of the roots of the equations of higher degree was precluded by the limited accuracy of the coefficients and independent of the method of solution.

The equation $x^{128} - 1 = 0$ was solved as an example of a special type of equation whose solution could be easily checked. The maximum error occurring in any root was of order 10^{-7} and the time for solution was 70 minutes.

No equation whose solution has been attempted has failed to yield convergence although as indicated in the table, the solutions of equations of large degree may be greatly in error. We conclude that convergence in the large does occur in most practical cases in spite of the fact that convergence has only been proved for single and double roots when the process has brought one to the neighborhood of a root.

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Everett's Formula for Bivariate Interpolation and Throwback of Fourth Differences

It is well known [1], [2], [6], [7], [8], [9], that it is possible in the case of univariate tables for use with Everett's formula, to eliminate columns of higher order differences with practically no loss of accuracy by modification of one or more lower order differences through a process known as throwback. That the same thing is possible with bivariate tables (and, presumably, with other multivariate tables) seems not to have been recorded in print.

Everett's formula for bivariate interpolation, as far as fourth order differences, can be written as follows, using symbolism similar to that of [2] (see also [3], p. 8–9 and [4], p. viii–xiv).

$$
(1) \quad
\begin{aligned}
u_{p,p'} &\equiv u(x_0 + ph, y_0 + p'k) \\
&= qq'u_{0,0} + qp'u_{0,1} + pq'u_{1,0} + pp'u_{1,1} \\
&\quad - E_2(q)\{q'\delta_x^2 u_{0,0} + p'\delta_x^2 u_{0,1}\} - E_2(p)\{q'\delta_x^2 u_{1,0} + p'\delta_x^2 u_{1,1}\} \\
&\quad - E_2(q')\{q\delta_y^2 u_{0,0} + p\delta_y^2 u_{1,0}\} - E_2(p')\{q\delta_y^2 u_{0,1} + p\delta_y^2 u_{1,1}\} \\
&\quad + E_4(q)\{q'\delta_x^4 u_{0,0} + p'\delta_x^4 u_{0,1}\} + E_4(p)\{q'\delta_x^4 u_{1,0} + p'\delta_x^4 u_{1,1}\} \\
&\quad + E_4(q')\{q\delta_y^4 u_{0,0} + p\delta_y^4 u_{1,0}\} + E_4(p')\{q\delta_y^4 u_{0,1} + p\delta_y^4 u_{1,1}\} \\
&\quad + E_2(q)E_2(q')\delta_x^2\delta_y^2 u_{0,0} + E_2(q)E_2(p')\delta_x^2\delta_y^2 u_{0,1} \\
&\quad + E_2(p)E_2(q')\delta_x^2\delta_y^2 u_{1,0} + E_2(p)E_2(p')\delta_x^2\delta_y^2 u_{1,1}
\end{aligned}
$$

where

$$p + q = 1 \quad \text{and} \quad p' + q' = 1$$

and where

$$E_2(x) = -\binom{x+1}{3}$$

and

$$E_4(x) = \binom{x+2}{5}.$$

We write the above as follows:

$$
(2) \quad
\begin{aligned}
u_{p,p'} &= qq'u_{0,0} + qp'u_{0,1} + pq'u_{1,0} + pp'u_{1,1} \\
&\quad - E_2(q)q'[\delta_x^2 u_{0,0} - c\delta_x^4 u_{0,0} - d\delta_x^2\delta_y^2 u_{0,0}] \\
&\quad - E_2(q')q[\delta_y^2 u_{0,0} - c'\delta_y^4 u_{0,0} - d'\delta_x^2\delta_y^2 u_{0,0}] \\
&\quad - E_2(q)p'[\delta_x^2 u_{0,1} - c\delta_x^4 u_{0,1} - d\delta_x^2\delta_y^2 u_{0,1}] \\
&\quad - E_2(p')q[\delta_y^2 u_{0,1} - c'\delta_y^4 u_{0,1} - d'\delta_x^2\delta_y^2 u_{0,1}] \\
&\quad - E_2(p)q'[\delta_x^2 u_{1,0} - c\delta_x^4 u_{1,0} - d\delta_x^2\delta_y^2 u_{1,0}] \\
&\quad - E_2(q')p[\delta_y^2 u_{1,0} - c\delta_y^4 u_{1,0} - d'\delta_x^2\delta_y^2 u_{1,0}] \\
&\quad - E_2(p)p'[\delta_x^2 u_{1,1} - c\delta_x^4 u_{1,1} - d\delta_x^2\delta_y^2 u_{1,1}] \\
&\quad - E_2(p')p[\delta_y^2 u_{1,1} - c'\delta_y^4 u_{1,1} - d'\delta_x^2\delta_y^2 u_{1,1}] + R,
\end{aligned}
$$

where c, c', d, and d' are constants to be selected so that $|R|$ is "small" for all p, p' on ranges $0 \leq p \leq 1$, $0 \leq p' \leq 1$, and where

$$
\begin{aligned}
(3) \quad R = & [E_4(q) - cE_2(q)]q'\delta_z{}^4 u_{0,0} + [E_4(q') - c'E_2(q')]q\delta_y{}^4 u_{0,0} \\
& \qquad\qquad + E_2(q)[E_2(q') - dq' - d'q]\delta_z{}^2\delta_y{}^2 u_{0,0} \\
& + [E_4(q) - cE_2(q)]p'\delta_z{}^4 u_{0,1} + [E_4(p') - c'E_2(p')]q\delta_y{}^4 u_{0,1} \\
& \qquad\qquad + E_2(q)[E_2(p') - dp' - d'q]\delta_z{}^2\delta_y{}^2 u_{0,1} \\
& + [E_4(p) - cE_2(p)]q'\delta_z{}^4 u_{1,0} + [E_4(q') - c'E_2(q')]p\delta_y{}^4 u_{1,0} \\
& \qquad\qquad + E_2(p)[E_2(q') - dq' - d'p]\delta_z{}^2\delta_y{}^2 u_{1,0} \\
& + [E_4(p) - cE_2(p)]p'\delta_z{}^4 u_{1,1} + [E_4(p') - c'E_2(p')]p\delta_y{}^4 u_{1,1} \\
& \qquad\qquad + E_2(p)[E_2(p') - dp' - d'p]\delta_z{}^2\delta_y{}^2 u_{1,1}.
\end{aligned}
$$

In equation (2) above, it is implied that the entire term in (1) involving $\delta_z{}^2 u_{i,j}$ is to be thrown back onto $\delta_z{}^2 u_{i,j}$, and that the entire term in (1) involving $\delta_y{}^4 u_{i,j}$ is to be thrown back onto $\delta_y{}^4 u_{i,j}$, while the single term in (1) involving $\delta_z{}^2\delta_y{}^2 u_{i,j}$ may be thrown back onto both $\delta_z{}^2 u_{i,j}$ and $\delta_y{}^2 u_{i,j}$. Let us define R_z as that part of R involving $\delta_z{}^4 u_{i,j}$, R_y as that part of R involving $\delta_y{}^4 u_{i,j}$, and R_{zy} as that part involving $\delta_z{}^2\delta_y{}^2 u_{i,j}$. Thus

$$(4) \qquad R = R_z + R_y + R_{zy}.$$

Since $\delta_z{}^4 u_{i,j}$, $\delta_y{}^4 u_{i,j}$ and $\delta_z{}^2\delta_y{}^2 u_{i,j}$ are essentially independent of each other, it seems natural to select c and c' so that $|R_z|$ and $|R_y|$ are small, and take care of R_{zy} later. This has the further advantage that it will take care of the special case where $\delta_z{}^2\delta_y{}^2 u$ is zero or nearly so. We do this by the method of [1] and [2]. In [2] it is shown (in effect) that if $c = c_0 = (3 + \sqrt{2})/24 \approx 0.1839256$, then the absolute value of

$$(5) \qquad R' = [E_4(p) - cE_2(p)]\delta^4 u_1 + [E_4(q) - cE_2(q)]\delta^4 u_0$$

is less than $L/2000$ provided $|\delta^4 u| < L$ and $|\delta^6 u| < 0.07L$. In equation (3) we have four groups of terms, two each in R_z and R_y, which resemble R'. For example, R_z can be written as

$$
\begin{aligned}
(6) \quad R_z = & q'\{[E_4(q) - cE_2(q)]\delta_z{}^4 u_{0,0} + [E_4(p) - cE_2(p)]\delta_z{}^4 u_{1,0}\} \\
& + p'\{[E_4(q) - cE_2(q)]\delta_z{}^4 u_{0,1} + [E_4(p) - cE_2(p)]\delta_z{}^4 u_{1,1}\}.
\end{aligned}
$$

The terms in braces are precisely analogous to the univariate expression R'.

Thus, if we select $c = c_0$, it follows immediately (since $q' + p' = 1$) that

$$(7) \qquad |R_z| < L/2000 \quad \text{provided} \quad |\delta_z{}^4 u| < L \quad \text{and} \quad |\delta_z{}^6 u| < 0.07L.$$

If we also select $c' = c_0$, we get, in a similar fashion

$$(8) \qquad |R_y| < M/2000 \quad \text{provided} \quad |\delta_y{}^4 u| < M \quad \text{and} \quad |\delta_y{}^6 u| < 0.07M.$$

Now R_{zy} consists of four terms, of which a typical one is

$$(9) \qquad \phi(q, q')\delta_z{}^2\delta_y{}^2 u_{0,0}.$$

where

$$\phi(q, q') = E_2(q)[E_2(q') - dq' - d'q].$$

The natural method of attack on the problem of minimizing the maximum absolute contribution of R_{xy} is to proceed in a manner similar to that of [2], viz., to write $\delta_x^2 \delta_y^2 u_{i,1} = \delta_x^2 \delta_y^2 u_{i,0} + \delta_x^2 \delta_y^3 u_{i,\frac{1}{2}}$, $i = 0$, 1. After these substitutions we can write R_{xy} as follows:

(10) $$R_{xy} = G\delta_x^2 \delta_y^2 u_{0,0} + H\delta_x^2 \delta_y^2 u_{1,0} + A\delta_x^2 \delta_y^3 u_{0,\frac{1}{2}} + B\delta_x^2 \delta_y^3 u_{1,\frac{1}{2}},$$

where

$$A = E_2(q)[E_2(p') - dp' - d'q],$$
$$B = E_2(p)[E_2(p') - dp' - d'p],$$
$$G = E_2(q)[E_2(q') + E_2(p') - d - 2d'q],$$
$$H = E_2(p)[E_2(q') + E_2(p') - d - 2d'p].$$

If we replace $\delta_x^2 \delta_y^2 u_{1,0}$ by $\delta_x^2 \delta_y^2 u_{0,0} + \delta_x^3 \delta_y^2 u_{\frac{1}{2},0}$ we can write above equation as

(11) $$R_{xy} = G^*\delta_x^2 \delta_y^2 u_{0,0} + H\delta_x^3 \delta_y^2 u_{\frac{1}{2},0} + A\delta_x^2 \delta_y^3 u_{0,\frac{1}{2}} + B\delta_x^2 \delta_y^3 u_{1,\frac{1}{2}}$$

where

$$G^* = G + H = [E_2(q') + E_2(p') - d][E_2(q) + E_2(p)]$$
$$- 2d'[qE_2(q) + pE_2(p)].$$

We assume, for the moment, that the three terms in (11) involving fifth order mixed differences are "small" in comparison with the fourth order term; we examine the problem of selecting d and d' so as to minimize the maximum absolute deviation from zero of the fourth order term in the region $D: 0 \le p \le 1$, $0 \le p' \le 1$.

Note that $G^*(p, p')$ is symmetric with respect to $p = 1/2$ and with respect to $p' = 1/2$. Therefore, we may restrict attention to the region $D^*: 0 \le p \le 1/2$, $0 \le p' \le 1/2$.

Evidently we may assume both $d > 0$ and $d' > 0$, since any positive extremum of G^* for $d \le 0$ could be made smaller by taking $d > 0$; the same is true of any positive extremum for $d' \le 0$.

We have

$$\partial G^*/\partial p = (1 - 2p)\{[E_2(p') + E_2(q') - d]/2 - 2d'[2p^2 - 2p + 1]/3\}$$
$$= -4d'(1 - 2p)(p - p_1)(p - p_2)/3$$

where

$$p_1 = 1/2 - \sqrt{g - 1/4}, \quad p_2 = 1/2 + \sqrt{g - 1/4},$$
$$g = \frac{3}{8d'}[E_2(q') + E_2(p') - d].$$

Also,

$$\partial G^*/\partial p' = (1 - 2p')[E_2(p) + E_2(q)]/2.$$

The extrema of G^* either lie on the boundary of D^* or satisfy the conditions $\partial G^*/\partial p = \partial G^*/\partial p' = 0$. The latter conditions imply that

$$\text{(a)} \quad p = 1/2, \quad p_1 \text{ or } p_2$$

and

$$\text{(b)} \quad p' = 1/2 \quad \text{or} \quad p = 0.$$

Since $G^*(0, p') \equiv 0$, the only non-zero extrema of G^* occur on one of the other three boundaries of D^*.

Along $p' = 0$, $g = -3d/8d' < 0$; hence both p_1 and p_2 are imaginary here, and therefore $\partial G^*/\partial p < 0$ on $0 \le p < 1/2$, $= 0$ at $p = 1/2$. Thus, $G^*(1/2, 0)$ $= -(d + d')/8 < 0$ is the minimum value of G^* on boundary $p' = 0$ of D^*.

Obviously, $\partial G^*/\partial p' > 0$ on the interior of D^* and along $p = 1/2$, and $\partial G^*/\partial p' = 0$ in D^* only along $p = 0$ and $p' = 1/2$. Hence, we can infer that $G^*(1/2, 0)$ is the absolute minimum of G^* in D^* and the only possible maximum is along $p' = 1/2$. We therefore next investigate G^* along $p' = 1/2$.

Along $p' = 1/2$, $g = 3(1/8 - d)/8d'$ and $g - 1/4 = (3 - 24d - 16d')/64d'$. It can be shown, by examination of $\partial G^*/\partial p$, that

a. if $3 - 24d - 16d' \le 0$, $G^*(1/2, 1/2)$ is a (negative) minimum and the absolute maximum of G^* on D^* is therefore 0.
b. if $3 - 24d - 16d' > 0$,
   and (1) $3 - 24d - 32d' < 0$, $G^*(1/2, 1/2)$ is a maximum, absolute if positive,
   and (2) $3 - 24d - 32d' \ge 0$, $G^*(1/2, 1/2) > 0$ and the absolute maximum of G^* on D^*.

In all cases, $G^*(1/2, 1/2) = 1/64 - (d + d')/8$.

Since we wish to minimize the maximum absolute deviation of G^* from 0, since $G^*(1/2, 1/2) - G^*(1/2, 0) = 1/64$, and since both $G^*(1/2, 1/2)$ and $G^*(1/2, 0)$ are monotonic increasing functions of $(d + d')$, we can evidently solve this minimax problem by making $G^*(1/2, 1/2) > 0$ and $= |G^*(1/2, 0)|$. From this it follows that $d + d' = 1/16$ and maximum absolute deviation of G^* from 0 is $1/128$. Note that the condition $d + d' = 1/16$ is consistent with case (b)(2) above.

Having determined that we should select $d + d' = 1/16$, whence $|G^*| \le 1/128$, we now consider the selection of d (and of $d' = 1/16 - d$) and its effect on fifth order terms in R_{xy}.

An investigation of the coefficients A, B, and H of the fifth order terms in R_{xy} by methods similar to the investigation of G^* shows that (when $d' = 1/16 - d$)

(1) Maximum $|H|$ is a monotonic decreasing function of d, $0 < d < 1/16$.
(2) Maximum $|A|$ and maximum $|B|$ are monotonic increasing functions of d on above range, and have same values, although attained for different values of (p, p').

Thus, if $|\delta_x^2 \delta_y^2 u| \le N$, we can say that

$$(12) \quad |R_{xy}| < N/128 + \max |H| \max |\delta_x^2 \delta_y^2 u| + 2 \max |A| \max |\delta_x^2 \delta_y^3 u|.$$

Unless we make some assumption about the relative magnitudes of $\delta_x^3\delta_y^2u$ and $\delta_x^2\delta_x^2u$, there is no rational basis for selection of the value of d other than a selection based on the fact that $\max|H|$ is monotonic increasing and $\max|A|$ is monotonic decreasing. For example, if we assume that $\max|\delta_x^3\delta_y^2u| = 2\max|\delta_x^2\delta_y^3u| \equiv P$, then we would be confronted with a problem of minimizing $P[\max|H| + \max|A|]$. If this were the case, it turns out we should choose d close to $1/32$ (midpoint of interval $0 < d < 1/16$) to minimize $[\max|H| + \max|A|]$. Since this choice at least has the virtue of symmetry (if $d = 1/32$, then $d' = 1/32$), we make it.

With the above choice, we find that

$$\max|H| < 0.00451$$

and

$$\max|A| < 0.00323.$$

However, maxima above are attained for different (p, p') from the pairs for which $|G^*|$ has its maximum, viz., $(1/2, 1/2)$ and $(1/2, 0)$. The maximum values of $|H|$ and $|A|$ for these pairs of arguments are respectively $1/256 \approx .00391$ and $1/512 \approx .00195$.

In order to determine a reasonable set of restrictions on $|\delta_x^2\delta_y^3u|$ and $|\delta_x^3\delta_y^2u|$, we remember that we already have assumed the fifth order terms "small" in comparison with the fourth order term whose maximum absolute value is $N/128$. If, e.g., $N \leq 60$ so that $N/128 \leq .46875$, and this term constitutes most of $|R_{xy}|$, then in (12) we may allow $\max|\delta_x^3\delta_y^2u| = \max|\delta_x^2\delta_y^3u| \leq 4$ and have $|R_{xy}| < 1/2$. Thus, if $|\delta_x^2\delta_y^2u| \leq N$ and $|\delta_x^3\delta_y^2u|$, $|\delta_x^2\delta_y^3u| \leq N/15$, then $|R_{xy}| < N/120$.

Obviously, other sets of restrictions are possible; e.g., we might allow the two fifth order differences different maxima, or we might allow them to be larger fractions of total $|R_{xy}|$. A similar result would be: if $|\delta_x^2\delta_y^2u| \leq N$ and $|\delta_x^3\delta_y^2u|$, $|\delta_x^2\delta_y^3u| \leq .28N$, then $|R_{xy}| < N/100$.

However, it seems natural to select a restriction set corresponding roughly to that chosen for $|\delta_x^4u|$ and $|\delta_x^5u|$ in [2], viz., L and $0.07L$ (thus, $1/15 \approx 0.07$).

In summary, then, we can say that with the choice $c = c' = c_0$ and $d = d' = 1/32$ for the "throwback" constants,

$$(13) \qquad |R| < L/2000 + M/2000 + N/120$$

provided

$$|\delta_x^4u| \leq L, \qquad |\delta_x^5u| \leq 0.07L,$$
$$|\delta_y^4u| \leq M, \qquad |\delta_y^5u| \leq 0.07M,$$
$$|\delta_x^2\delta_y^2u| \leq N, \quad |\delta_x^3\delta_y^2u| \lesssim N/15,$$

and

$$|\delta_x^2\delta_y^3u| \leq N/15.$$

In some instances, it may be desirable to have available the general result corresponding to (13), which does not necessarily assume that the fifth order differences are "small" with respect to the fourth order terms.

In order to see how we get such a result, consider, as in [2], that we have written equation (5) as follows:

$$(5') \qquad R' = G\delta^4u_0 + [E_4(p) - cE_2(p)]\delta^5u_{\frac{1}{2}}$$

where

$$G \equiv p(p^2 - 1)(p - 2)/24 + cp(p - 1)/2.$$

With the choice $c = c_0$ (which minimizes maximum $|G|$), we find that $|G| < .000447$ and $|E_4(p) - cE_2(p)| < .000792$. Consequently, if $|\delta^4 u_0| = 1000$ and $|\delta^4 u_1| < 70$, we would have $|R'| < .503$, which is approximately $1/2$. Indeed, if we make use of the fact that $\max |G|$ and $\max |E_4(p) - cE_2(p)|$ occur for different values of p, and that at $p = .88265$, $|G| = \max$, whereas $|E_4(p) - cE_2(p)| \approx .000744$ instead of $.000792$, we find that, probably, $|R'| < .499$. From this *arbitrary* selection of $\max |\delta^4 u| = 1000$ comes, then, the result stated in connection with equation (5). Evidently, the corresponding general (and *conservative*) result is that if $|\delta^4 u| \leq L$ and $|\delta^6 u| \leq KL$, then

$$(5'') \qquad\qquad |R'| < L(.000447 + .000792K).$$

Obviously, this implies that

$$(7') \quad |R_x| < L(.000447 + .000792K_1) \quad \text{if} \quad |\delta_x^4 u| \leq L \quad \text{and} \quad |\delta_x^6 u| \leq K_1 L.$$

Similarly, we have

$$(8') \quad |R_x| < M(.000447 + .000792K_2) \quad \text{if} \quad |\delta_y^4 u| \leq M \quad \text{and} \quad |\delta_y^6 u| \leq K_2 M.$$

(In the above, the only restrictions on the K_i are that they be > 0.)

From the discussion in connection with equation (12) above, we see that we have

$$(12') \qquad\qquad |R_{xy}| < N[1/128 + .00451K_{32} + .00646K_{23}]$$

if

$$|\delta_x^2 \delta_y^2 u| \leq N, \quad |\delta_x^3 \delta_y^2 u| \leq K_{32} N,$$

and

$$|\delta_x^2 \delta_y^3 u| \leq K_{23} N,$$

where, again, the only condition on the K_{ij} is that they be positive.

Consequently, we have the following general result to replace (13):

$$(13') \quad |R| < L(.000447 + .000792K_1) + M(.000447 + .000792K_2) \\ + N(1/128 + .00451K_{32} + .00646K_{23})$$

under the hypotheses governing (7'), (8'), (12').

The advantages of the use of modified second differences as in equation (2) are obvious—a published table need contain no fourth order differences and formula (2) requires only 24 multiplications to evaluate 12 terms as opposed to the 48 multiplications required to evaluate 24 terms in formula (1).

It should be noted that the fraction $1/120$ is disappointingly large in comparison with $1/2000$. However, there really seems to be nothing one can do about this situation. Even so, we have "gained" in some sense, since the condition for negligibility of terms involving fourth order (pure and mixed) differences is that

$$(14) \quad L/42 + M/42 + N/64 < 1/2 \quad \text{(units of last decimal place tabulated)}$$

whereas the condition for "throwback ability" of the same differences is

(15) $L/2000 + M/2000 + N/120 < 1/2,$

in the presence of auxiliary conditions in (13).

As an example of the use of throwback, we may consider its application to the tables in [5]. Cursory examination of [5] indicates that in a large proportion of the table, inequality (15) (or its more general version corresponding to (13')) is satisfied, so that modified second differences could be made to work. As a specific illustration, we take the numerical example on p. xxiv–xxv, viz., computation of $I(4.025, 7.05)$. We use formula (2) as modified by introduction of the throwback constants c, c', d, d' selected as above, viz.,

$$
\begin{aligned}
(2') \quad u_{p, p'} &\approx qq'u_{0,0} + qp'u_{0,1} + pq'u_{1,0} + pp'u_{1,1} \\
&- E_2(q)q'M_z^2u_{0,0} - E_2(q)p'M_z^2u_{0,1} - E_2(p)q'M_z^2u_{1,0} - E_2(p)p'M_z^2u_{1,1} \\
&- E_2(q')qM_y^2u_{0,0} - E_2(p')qM_y^2u_{0,1} - E_2(q')pM_y^2u_{1,0} - E_2(p')pM_y^2u_{1,1}
\end{aligned}
$$

where $M_z^2u_{i,j}$ is the modified second difference of u, with respect to the variable z, evaluated at "point" i, j.

At the point in question, $I(4.0, 7.0) \sim u_{0,0}, I(4.1, 7.0) \sim u_{1,0}, I(4.0, 7.2) \sim u_{0,1},$ and $I(4.1, 7.2) \sim u_{1,1}$. Thus, the full array of differences for use in the unmodified formula would be:

$$
\begin{array}{cc}
\delta_z^2 & \delta_y^2 \\
\delta_z^4 & \delta_y^4 \\
\multicolumn{2}{c}{\delta_z^2\delta_y^2}
\end{array}
$$

(0, 0)			(0, 1)		
	-17520	-1747		-17985	-1745
	63	0		80	0
		16			17
(1, 0)			(1, 1)		
	-16319	-1643		-16802	-1649
	30	0		43	0
		10			13

This is replaced by the following set of modified differences:

$$
\begin{array}{cc}
M_z^2 & M_y^2
\end{array}
$$

(0, 0)		(0, 1)	
	-17532 -1748		-18000 -1746
(1, 0)		(1, 1)	
	-16325 -1643		-16810 01649

Here we have $p = 0.25$, $p' = 0.25$. Using (2'), together with a table of $E_2(x)$, as in [2], we get $I(4.025, 7.05) \approx .878, 5410$, which agrees with result obtained by Pearson.

In principle, throwback of other kinds in the bivariate Everett formula is possible. There is, also, no a priori reason why it could not be accomplished in the trivariate Everett formula, which is stated in [4], page x.

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TECHNICAL NOTES AND SHORT PAPERS

On the Treatment of Monte Carlo Methods in Text Books

Though Monte Carlo methods have not yet reached the text book stage, they are gradually being afforded a brief mention in new books on numerical analysis. This is probably stimulated by a desire for completeness but the brevity of treatment often results in a very narrow picture of Monte Carlo. For any given problem there may be several methods of solution by Monte Carlo and there are general principles which should guide one in choosing between these. A description of a single method, leaving the reader with the impression that this is the sole method, is therefore a mistake. Care must also be taken to see that, though a method may be illustrated on a simple example for which Monte Carlo would not normally be used, it is easily extensible to problems for which Monte Carlo would be needed and provides a reasonably practical method of solving these.

In two otherwise excellent books by Householder [2] and Kopal [4] a technique is given for evaluating integrals by Monte Carlo. It is the only example of Monte Carlo in either book and is rather an unfortunate one; for it is not easily applicable in practice to multiple integrals though it is largely for these that Monte Carlo can be useful; in addition, it is always less efficient, and often much less so, than another well-known simple technique (defined as crude Monte Carlo by Hammersley and Morton [1]). Their comparative efficiency is also an admirable illustration of a general precept to be used when choosing even the simplest Monte Carlo methods.

Consider the estimation of the integral

$$I = \int_0^1 f(x)dx$$

where $f(x)$ is a prescribed function whose minimum is made zero by a change of origin and whose maximum is a positive constant M. In the method described in [2] and [4], we draw a rectangle of height M and unit breadth about the curve $y = f(x)$ and choose n points $(\xi_i, M\eta_i)$, $i = 1, 2, \cdots, n$, at random in this rectangle; ξ_i and η_i are independent random numbers uniformly distributed in the interval $(0, 1)$. We count a score $t_{1i} \equiv t_1(\xi_i, \eta_i)$ for each point, t_{1i} being M if the point lies between $y = f(x)$ and the x-axis and zero otherwise. Then the mean score $t_1 = (1/n) \sum_{i=1}^n t_{1i}$ is an unbiased estimator of I and t_1/M is binomially distributed with parameter I/M, i.e.,

$$\epsilon(t_1) = I \quad \text{and} \quad \text{var } t_1 = I(M - I)/n.$$

The crude Monte Carlo technique, first described by Kahn [3] in 1949, consists of using $t_2 = f(\xi)$ as an estimator of I. Thus, for the n random numbers ξ_i, we have a mean score $t_2 = (1/n) \sum_{i=1}^n f(\xi_i)$ for which clearly

$$\epsilon(t_2) = I \quad \text{and} \quad \text{var } t_2 = (1/n) \left\{ \int_0^1 [f(x)]^2 \, dx - I^2 \right\} = (S^2 - I^2)/n.$$

It is clear that, when dealing with multiple integrals, the bounds of the integrand which it is necessary to know for the first method may be very difficult to obtain even when finite; and that any overestimate of them can lead to a considerable increase in var t_1. Moreover, since $f(x) \geqslant 0$, $S^2 \leqslant IM$ so that var $t_2 \leqslant$ var t_1, the equality occurring only when $f(x)$ is a constant; the importance of this difference is enhanced by the fact that it is the ratio of the variances which measures the relative efficiency of the two methods. For example, when $f(x) \equiv x$, $IM = 1/2$ and $S^2 = 1/3$ but var $t_1 = 1/4 = 3$ var t_2, so that the second method is three times as efficient as the first. Finally, it should be noted that $t_2 = \int_0^1 t_1(\xi, \eta)d\eta$, i.e., M times the probability that, for given ξ, the point $(\xi, M\eta)$ will lie below $y = f(x)$: thus the second estimator is simply related to the first, being its average over η, and its higher efficiency is due to following the principle that random processes should be replaced by analysis wherever possible, as pointed out in [1].

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1. J. M. HAMMERSLEY & K. W. MORTON, "A new Monte Carlo technique: antithetic variates," Cambridge Phil. Soc., Proc., v. 52, 1956, p. 449–475.
2. A. S. HOUSEHOLDER, Principles of Numerical Analysis, McGraw-Hill Book Co., Inc., New York, 1953, p. 243.
3. H. KAHN, "A modification of the Monte Carlo method," Seminar on Scientific Computation, Proc., I.B.M., New York, 1949, p. 20–27.
4. Z. KOPAL, Numerical Analysis, Chapman and Hall Ltd., London, 1955, p. 428–429.

On the Simultaneous Numerical Inversion of a Matrix and All its Leading Submatrices

This note considers the case in which a square matrix A may be factorized

$$(1) \qquad A = X'DY,$$

where X and Y are square matrices whose diagonal elements (where $i = j$) are unity, and whose subdiagonal elements (where $i > j$) are zero, whilst D is a diagonal matrix, that is, one whose off-diagonal elements (where $i \neq j$) are zero [1]. Where this factorization exists, it is unique [1]. Where it does not exist, there exists another factorization of the form (1) in which the matrices X and Y differ from the foregoing specification by having their rows in some way interchanged [2].

Let the matrices appearing in relation (1) all be partitioned after the rth row and column, the resultant leading submatrices of order r by r being denoted by A_r, X_r', D_r, Y_r, where $r = 1, 2, \cdots, n - 1$. Then it is easily seen that

$$A_r = X_r'D_rY_r.$$

If D_r is non-singular, so is A_r, and hence

$$(2) \qquad A_r^{-1} = Y_r^{-1}D_r^{-1}X_r'^{-1}.$$

This holds also for $r = n$, when suffixes may be omitted. It is, in fact, equivalent to taking $r = n$ and replacing the elements in the last $n - r$ columns and rows of all matrices by zero. This suggests that if a typical element, say b_{ij}, of $A^{-1} = B$ is obtained as the scalar product of the ith row of $Y^{-1}D^{-1}$ and the jth row of X^{-1}, the corresponding element $b_{ij(r)}$ (where $i, j \leqslant r$) of $A_r^{-1} = B_r$, say, is obtained as the rth partial sum of the n products of the row-by-row multiplication. Thus, with a desk calculating machine, on which products can be accumulated, the inverses $B_1, B_2, \cdots,$ of $A_1, A_2, \cdots$ are obtained simply as the partial sums of the row-by-row multiplications needed anyhow for the computation of the inverse B of A.

As the factorization is unique any process yielding X', D, and Y (or $Y^{-1}D^{-1}$ and X'^{-1}, etc.) may be used. The following arithmetic check is suggested. Relation (1) gives

$$X'^{-1}A = DY.$$

Comparison of diagonal elements on either side shows that the scalar product of the rth column of X_r^{-1} into the rth column of A_r is equal to d_r, the rth diagonal element of D_r. This relation may be used to check each column of X^{-1} fresh after computation. Similarly it is found that the scalar product of the rth column of Y_r^{-1} into the rth row of A_r is also equal to d_r.

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1. A. M. TURING, "Rounding-off errors in matrix processes," *Quart. Jour. Mech. and Appl. Math.*, v. 1, 1948, p. 289.
2. R. A. FRAZER, W. J. DUNCAN, & A. R. COLLAR, *Elementary Matrices and Some Applications to Dynamics and Differential Equations*, Cambridge Univ. Press, 1938, p. 99.

A Note on Large Linear Systems

Towards the end of 1955 the author of the present note was charged by Svenska Aeroplan Aktiebolaget (SAAB Aircraft Company), with setting up a program for the automatic computer BESK for the solution of large symmetric systems of linear algebraic equations. The reason was that the solution of 4 systems with 214 unknowns was required. The systems had been set up in conjunction with certain calculations for airplane structural analysis. Approximately 40% of the coefficients in the matrices of these systems differed from zero. This fact was taken into account in setting up the program. Nevertheless, the whole system could not be accommodated at one time in the magnetic drum memory of 8192 words associated with BESK, and a splicing method was developed which does not involve excessive loss of time in connection with the input and output of intermediate results. The four systems were solved simultaneously by the BESK. The method used for solving the equations was Gauss' method of elimination. The calculating time was about 2 hours. If $p\%$ of the coefficients of the systems differ from zero, and the order of the systems is n, the calculating time will be proportional to $p^2 n^3$.

By formation of residuals and one iteration a solution was obtained which, when inserted into the original systems, proved to satisfy the equations with deviations less than $10^{-60}\%$, i.e., if the computed solution was $\bar{x}_j$ and the system was $AX = B$, the "error vector" $E = \sum_j a_{ij}\bar{x}_j - b_i$ satisfied $|E| < 10^{-8}|B|$. The systems were badly conditioned since, if the matrix was (a_{ik}), the "condition number" was:

$$R(A) = \frac{|\text{determinant } (a_{ik})|}{\prod_{k=1}^{214} \left(\sum_{i=1}^{214} |a_{ik}| \right)} = 2^{-856}.$$

The matrix is therefore comparable with the $n \times n$ matrix

$$A_n = \begin{bmatrix}
5 & -4 & 1 & \cdot & \cdot & \cdot & \cdot & & & \\
-4 & 6 & -4 & 1 & \cdot & \cdot & \cdot & & & \\
1 & -4 & 6 & -4 & 1 & \cdot & \cdot & & & \\
 & 1 & -4 & 6 & -4 & 1 & \cdot & & & \\
 & & & & \cdot & & & & & \\
 & & & & & \cdot & & & & \\
 & & & & & & \cdot & & & \\
 & & & 1 & -4 & 6 & -4 & 1 & & \\
 & & & \cdot & 1 & -4 & 6 & -4 & & \\
 & & & \cdot & \cdot & 1 & -4 & 5 & &
\end{bmatrix}$$

studied by Todd, for which $R(A_n) \sim c \cdot n^2 \cdot 2^{-4n}$ for a certain constant c. No direct comparison of the results of our experiments with those made by Todd [1] (for a 49 × 49 matrix, without iteration) is possible, since his results are expressed in terms of the norm of the error matrix.

The calculating program thus described operates at a fixed binary point and uses 40 binary digits. A program operating at a floating binary point and including 32 binary digits in its digital portion and 8 binary digits in its scalar exponent has also been developed. This program works at an approximately 30% slower rate and gave, when applied to the above-mentioned systems, a slightly better solution.

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1. J. Todd, "The condition of certain matrices II," *Arch. d. Math.*, v. 5, 1954, p. 249–257

REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

65[A].—ROBERT L. CAUSEY, *Decimal to Octal and Octal to Decimal Conversion Tables*, U. S. Naval Air Missile Test Center, Point Mugu, California, 1952, 30 p., 26.5 cm. Deposited in the UMT FILE.

These are radix tables with three figure groups; when using the decimal to octal table the tabular entries are added together using octal arithmetic. The decimal to octal table has 11 place octal values of $n \times 10^{-K}$ for $n = 0(1)999$ and $K = 3, 6, 9$. The octal to decimal table has 10 place decimal values of $n \times 8^{-K}$ for $n = 0(1)511$ and $K = 3, 6, 9, 12$, together with an argument column giving $n \times 8^{-3}$ in octal. The tables were printed by photo-offset from output of an IBM tabulator. This machine has the unpleasant characteristic of leaving more space between adjacent columns than rows. The tables were developed for use with the RAYDAC.

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66[D, E].—HARVARD UNIVERSITY, Computation Laboratory, *Annals*, *Tables of the Function arc sin z*, v. 40, Harvard University Press, Cambridge, Mass., 1956, xxxviii + 586 p., 27 cm. Price $12.50.

This table lists values of $w =$ arc sin z, both the function w and the argument z being expressed in cartesian form. Symmetries make it unnecessary to consider values of z belonging to points outside the first quadrant. If $w = u + iv$ and $z = x + iy$, both u and v are tabulated to 6 decimals for various values of x and y. The results are arranged in 11 tables, in every one of which the lattice is square, that is, the intervals in x and y are equal. The intervals $\delta = \Delta x = \Delta y$ in Tables I to XI are respectively 0.002, 0.005, 0.01, 0.02, 0.05, 0.1, 0.2, 0.5, 1, 2, 5. In a table of a function of a complex variable, the earlier intervals are strikingly small; previous tables of sin z and arc sin z have rarely used intervals finer than

0.02. The regions in which the various intervals apply are clearly set out in detail on the contents page, and need not be reproduced here. It seems sufficient to state that

(i) the smallest intervals relate to the regions which keep closest to the branch point at $z = 1$;

(ii) the region covered by Table I ($\delta = 0.002$) is the rectangle $0.916 \leq x \leq 1.090, 0 \leq y \leq 0.140$;

(iii) the other regions are not single rectangles, but are each composed of two or three rectangles, the maximum width of the composite region varying in x from 77δ to 139δ, and in y from 95δ to 105δ;

(iv) the whole region covered in the volume is the square

$$0 \leq x \leq 475, \quad 0 \leq y \leq 475,$$

minus that part of the square for which both $x > 195$ and $y > 350$.

In general, interpolation is by Taylor's series, using the first two derivatives; the quantities actually tabulated are not these derivatives, $f'(z)$ and $f''(z)$, but the "pseudo-differences" $\delta f'(z)$ and $\delta^2 f''(z)/2$, of which the real and imaginary parts are given. (The reviewer regrets the use of the symbol δ to denote width of interval; he would have preferred to call it almost anything else, for example h.) Where use of two derivatives does not suffice, reference is given to the appropriate procedure—usually the employment of an earlier (finer mesh) table, but in the case of a region of Table I near to $z = 1$, the employment of an auxiliary function tabulated on pages xxxii and xxxiii.

It is interesting to see what an extensive table of a function of a complex variable can be produced in a fairly routine fashion with modern equipment. The present table was produced by the Computation Laboratory for the Aeronautical Research Laboratory of the United States Air Force, and its extent was presumably fixed by the requirements of that institution. Previous tables of both the direct and the inverse sine of complex argument have been mostly the work of electrical engineers. As the authors point out, the real and imaginary parts of the direct function sin z are given so simply by the expressions sin x cosh y and cos x sinh y that tabulation seems unnecessary, though the importance of the function has caused a number of tables to be made. In contrast, the computation of the inverse function arc sin z from ordinary tables of functions of one real variable is rather tedious, yet, as the authors rightly state, not much tabulation of the inverse function has been undertaken; useful tables listing a cartesian function of a polar argument were computed in the 1920's, the moving spirit being F. Emde. Concerning the references on page xxiv, it may be added that an abridgement of Hawelka's table has been printed by Boll [1], and that the small table given in the second edition of Jahnke and Emde was later transferred to Emde's volume of tables of elementary functions [2]. As the present tables list a cartesian function of a cartesian argument, they break new ground as regards form, besides being of record extent. They are manifestly to be welcomed for their own sake. They could also provide material if it should ever be desired to produce a smaller and cheaper table, such as would satisfy the needs of many. The reviewer ven-

tures this suggestion in case any persons should feel that a little acorn planted by Emde has grown into a splendid oak which is too big for their garden.

A. F.

1. MARCELL BOLL, *Tables Numériques Universelles des Laboratoires et Bureaux d'Études*, Dunod, Paris, 1947, p. 732 (see *MTAC*, v. 2, 1947, p. 336).
2. F. EMDE, *Tafeln elementarer Funktionen*, Leipzig and Berlin, 1940, second edition, Leipzig, 1949, p. 138 (see *MTAC*, v. 1, 1945, p. 384 and v. 4, 1950, p. 79). There are American reprints of both the second edition of Jahnke and Emde and the first edition of Emde.

67[F].—R. J. PORTER, "On irregular negative determinants of exponent $9n$," *MTAC*, v. 10, 1956, p. 22–25.

The determinant $D = b^2 - ac$ of the quadratic form $ax^2 + 2bxy + cy^2$ is called irregular if the principal genus is non-cyclic. The number of classes divided by the highest period is called the exponent of D. The first 58 values of $-D$ with exponents $9n$ are listed, giving the complete table up to 150,000. Tables are given which reveal special properties and a method for squaring a class of forms is outlined. The discriminant -3299 seems to play an exceptional role in various respects.

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This review was prepared by O. Taussky-Todd for *Mathematical Reviews*.

68[F, K, W].—GÖSTA NEOVIUS, "Artificial traffic trials using digital computers," Chapter 2. Generating pseudo-random numbers, *Ericcson Technics* 2, 1955, p. 279–291.

Five methods of generating pseudo-random numbers to radix 16 are described. The first method produces the Fibonacci series 1, 1, 2, 3, 5, 8, 13, $\cdots$ reduced modulo 2^{40} and the others are rather complicated modifications of this method which, however, are also convenient for the electronic computer BESK. For a description of the mathematical properties of Fibonacci sequences modularly reduced, see Duparc, Lekkerkerker, and Peremans [1]. The four modified Fibonacci sequences, but not the unmodified sequence, were found to satisfy the four tests for randomness described by M. G. Kendall and B. Babington Smith [2]. It was in a sense fortunate that the radix used was large (16) instead of 2; otherwise the serial test would certainly have led to the rejection of all the methods, unless this test had been interpreted in accordance with the reviewer's modification [3] of the analysis in [2].

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1. H. J. A. DUPARC, C. G. LEKKERKERKER, & W. PEREMANS, *Reduced sequences of integers and pseudo-random numbers*, Math. Centrum, Amsterdam, *Report* ZW 1953–002 (also *Report* ZW 1952–013).
2. M. G. KENDALL & B. BABINGTON SMITH, *Tables of Random Sampling Numbers*, Tracts for Computers, XXIV, Cambridge University Press, 1951.
3. I. J. GOOD, "The serial test for sampling numbers and other tests for randomness" Camb. Phil. Soc., *Proc.*, 49, 1953, p. 276–284.

69[G, H, X].—MARK LOTKIN, "A set of test matrices," *MTAC*, v. 9, 1955, p. 153–161.

The finite Hilbert matrix $B_n = ((i + k - 1)^{-1})$, $i, k = 1, 2, \cdots, n$ is often used for testing numerical methods in matrix computations. In order to have test data available for a non-symmetric matrix a study and tabulations are carried out for $A_n = (a_{ik})$ with $a_{1k} = 1, k = 1, \cdots, n$ and $a_{ik} = (i + k - 1)^{-1}$ for $i = 2, \cdots, n, k = 1, \cdots, n$. It is known that the det B_n is the reciprocal of a positive integer and it is now shown that det A_n is $(-1)^{n-1}$ times the reciprocal of a positive integer. The elements of A_n^{-1} are integers, which is also known to be true for B_n^{-1}. While all the characteristic roots of B_n are positive the tabulations indicate that for A_n, apart from the dominant characteristic root, all others have negative signs. Estimates for the characteristic root and vectors are studied as well as the M- and P-condition numbers (see John Todd, "The condition of a certain matrix," Cambridge Phil. Soc., *Proc.*, v. 46, 1950, p. 116–118). The paper includes tables for $n = 1$ to 10 of det A_n, the M- and P-condition numbers, the characteristic roots of largest and smallest absolute value and the corresponding vectors; A_6^{-1} and $H_6 = (A_6)^T A_6$ are also given. Results are given exactly, or to about 7S.

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This review was prepared by O. Taussky-Todd for *Mathematical Reviews*.

70[G, K].—F. N. DAVID & M. G. KENDALL, "Tables of symmetric functions, Part V," *Biometrika*, v. 42, 1955, p. 223–243.

These tables complete the services by the authors [1] in which each of the 4 kinds of fundamental symmetric functions, monomial, one-part, unitary, and homogeneous product sums are expressed in terms of each of the others for weights up to and including 12. The present tables are for the one-part, unitary pair. The table for the one-part, homogeneous product sum pair would be the same except for obvious changes in sign.

C. C. C.

1. F. N. DAVID & M. G. KENDALL, "Tables of symmetric functions Part I; Parts II, and III"; Part IV, *Biometrika*, v. 36, 1949, p. 431–449; v. 38, 1951, p. 435–462; v. 40, 1953, p. 429–446. [*MTAC*, v. 4, p. 146, v. 6, p. 224–225; see also corrigendum, v. 8, p. 188, v. 8, p. 150.]

71[I, L, P].—ROSS E. GRAVES, "Design and Analysis of an Optimum Comb Filter Based upon an Information Storage Device," GERA-20, Goodyear Aircraft Corp., Litchfield Park, Arizona, 1953, ii + 65 p.; "Miscellaneous Considerations in Comb Filter Design," GERA-24, Goodyear Aircraft Corp., Litchfield Park, Arizona, 1954, v + 74 p., photostat copies in notebook binders, 28 cm. Both copies deposited in the UMT FILE.

These papers contain, among other things, several devices for computation of the coefficients in the (finite) Fourier series for the function $T_{n-1}(Z_0 \cos u)$, where $T_K(x)$ is the Chebyshev polynomial of degree K, defined by $\cos (K \arccos x)$.

GERA-20 has two recursion formulas for computing the above coefficients;

numerical examples of their use for $n = 40$ relations between the coefficients and the hypergeometric function, plus relations between the coefficients and the Jacobi polynomial, are given in the Addenda.

GERA-24 includes "An Exact Method for Computation of the Tchebycheff Coefficients." The coefficients referred to are proportional to the above coefficients (in GERA-20), the so-called weights for an n-signal Chebyshev filter. The formula given is stated to be well suited for computation.

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72[I, L, P].—ISABELLE ARSHAM, *Chebyshev Coefficients for Chebyshev Polynomials* of *Orders 12 and 24 under the General Linear Transformation* (*U*), Report No. TR-326, Diamond Ordnance Fuze Laboratories, Ordnance Corps, Department of the Army, Washington, D. C., 1956, 27 p., 26.6 cm.

Consider the equation $T_n(ax + b) = \sum_{m=0}^{n} b_m T_m(x)$, where $T_m(x)$ is the Chebyshev polynomial of degree m.

This paper lists the coefficients $b_m (m = 0, 1, \cdots, n)$ as functions of a and b, for the cases $n = 12$ and 24.

The special case in which $b = 0$ is related to the problem considered in GERA-20 and GERA-24 (see preceding review 71). The b_m are needed in such situations as antenna design.

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73[K].—NBS Applied Mathematics Series, No. 21, *A Guide to Tables of the Normal Probability Integral*, U. S. Gov. Printing Office, Washington, D. C., 1952, iv + 16 p., 26.0 cm. Price $0.15.

This guide is not meant to be either an exhaustive or historical compilation but rather to assist a researcher or student to find the normal probability table he needs in standard texts and other sources likely to be available to him. The Guide is in two main portions. In the first, part I lists tables of segments of area under the normal curve corresponding to the abscissa x of five kinds: from $-x$ to x, from 0 to x, exterior of the interval, $(-x, x)$, from x to ∞, and from $-\infty$ to x. Part II contains list of tables inverse to those in part I, i.e., for given areas of any of the five kinds, the corresponding x is tabulated. Each set of listings is headed by a sketch of the area in question, the formula for it, and a key showing a few illustrative tabled values. The actual listings contain for each table: (i) the number of decimals or significant figures, (ii) intervals and ranges of the argument, (iii) indication of the differences, derivatives, or proportional parts given, and (iv) authorships, date, and page or table number. Part III gives descriptions of appropriate interpolation methods and Part IV gives relationships between some types of normal tables and other tabulated functions. The second main portion is the bibliography of tables arranged by author and year for reference from column (iv) of the listings in parts I and II. The arrangement and notation

follows that in Chapter 15 of A. Fletcher, J. C. P. Miller, & L. Rosenhead, *An Index of Mathematical Tables*, McGraw-Hill Book Co., New York, 1946 from which much of the content is taken. This Guide will be very useful and should adequately meet the ordinary needs of the researcher or student.

C. C. C.

74[K].—BIOMETRIKA, Editors, Editorial, "The normal probability function: Tables of certain area-ordinate ratios and of their reciprocals," *Biometrika*, v. 42, 1955, p. 217–222.

In distribution problems such as that of correlation between two continuous variables when one is dichotomized or in the case of percentile points one frequently is interested in the ratio between the area under the normal distribution to the left or right of a given deviate and the corresponding ordinate. The tables given in this editorial give values of the four ratios P/Z, Q/Z, Z/P and Z/Q for $X = 0(.01)3$. Here

$$Z = Z(X) = \frac{1}{\sqrt{2\pi}} e^{-X^2/2},$$

$$P = P(X) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-v^2/2} dv, \quad Q = 1 - P = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-v^2/2} dv.$$

The ratios P/Z, Q/Z and Z/Q are tabled to 5D and those for Z/P to 5S.

These tables form useful companions for Table II, *Tables for Statisticians and Biometricians*, Part II, which give the same four ratios for an argument of P instead of X. Both tables will be included in the second volume of *Biometrika Tables for Statisticians*.

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75[K].—D. TEICHROEW, "Numerical Analysis Research unpublished statistical tables," Amer. Stat. Assn., *Jn.*, v. 50, 1955, p. 550–556.

This note describes tables connected with probability and statistics which were computed at what is now Numerical Analysis Research, University of California at Los Angeles. It is likely that many of these tables will not be published, and most of them exist on punched cards. The tables are listed and individually summarized under 5 categories:

I. Tables associated with the normal distribution.
   Most of these tables are of functions useful in connection with computing moments of order statistics.
II. Tables associated with the gamma distribution.
   These are tables used in the computation of probability points of χ^2.
III. Tables associated with the t-distribution.
   These are tables used in computation of probability points of t.
IV. Tables for selecting samples from certain distributions

V. Miscellaneous tables.
 1. Multinomial coefficients
 2. Coefficients for curve fittings by Chebyshev polynomials
 3. Tables for Probit Analysis with Poisson error models.

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76[K].—A. A. ANIS, "The variance of the maximum of partial sums of a finite number of independent normal variates," *Biometrika*, v. 42, 1955, p. 96–101.

Let $x_1, x_2, \cdots, x_n$ be independent standard normal variates, let $S_r = x_1 + x_2 + \cdots + x_r$ and let $U_n = \max \{S_1, S_2, \cdots S_n\}$. It was shown by Anis and Lloyd [1] that $E(U_n) = \sum_{r=1}^{n-1} r^{-\frac{1}{2}}/\sqrt{2\pi}$ which is known to be $[2\sqrt{n-1} + \zeta(\frac{1}{2}) + 0(1/\sqrt{n})]/\sqrt{2\pi}$. In the present paper it is shown that $E(U_n^2) = (n+1)/2 + \sum_{r=1}^{n-2} \sum_{s=1}^{r}[s(r-s+1)]^{-\frac{1}{2}}/2\pi$. There is a table of $E(U_n)$, $E(U_n^2)$ and $\sigma(U_n)$ to 4D for $n = 3(1)25$.

Anis approximates the double sum by a double integral in order to write $E(U_n^2) \sim n - \sqrt{n}(2 + \sqrt{2})/\pi$ to terms of order $\sqrt{n}$. This formula, together with that for $E(U_n)$ mentioned above, implies $\sigma(U_n) \sim \sqrt{(\pi-2)/\pi}\sqrt{n} - [2 + \sqrt{2} + 2\zeta(\frac{1}{2})]/2\sqrt{\pi(\pi-2)}$ which does not agree well with the tabled values. Using another method, the reviewer finds $\sqrt{(n-1)/1} + \sqrt{(n-2)/2} + \cdots + \sqrt{1/n-1} \sim \pi n/2 + \zeta(\frac{1}{2})\sqrt{n}$ with the aid of which the double sum may be evaluated. We get $E(U_n^2) \sim n + 2\zeta(\frac{1}{2})\sqrt{n}/\pi$ and hence find constant term 0 in the expression for $\sigma(U_n)$. This is in agreement with the tabled values, which exceed $0.60281\sqrt{n}$ by about $0.478/\sqrt{n}$.

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1. A. A. ANIS & E. H. LLOYD, "On the range of partial sums of a finite number of independent normal variates", *Biometrika*, v. 40, 1953, p. 35–42.

77[K].—JOSEPH BERKSON, "Estimate of the integrated normal curve by minimum normit chi-square with particular reference to bio-assay," Amer. Stat. Assn., *Jn.*, v. 50, 1955, p. 529–549.

This article is the latest of a series of discussions of the relative merits of various methods of estimating the parameters (α and β) of the straight line transform of the integrated normal curve, used to a great extent in bio-assay. The work "normit" is coined to represent a normal deviate, i.e.,

$$\nu_i = \text{normit } P_i = \alpha + \beta_i,$$

where

$$P_i = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha+\beta x_i} e^{-x^2/2}dx,$$

and x_i is a function of the dosage producing P_i kills.

The author favors the minimum normit χ^2 method over that of maximum likelihood to estimate α and β from a set of x_i and p_i (estimate of P_i). He substantiates this preference with the results of a sampling experiment consisting of 600 samples for each of four sets of P's, each set consisting of P_1 (low dose), P_2 (mid dose) and P_3 (high dose).

He considers both estimation of α for β known and estimation of both α and β. In all cases the mean square error for the minimum normit χ^2 method was less than for the maximum likelihood method.

Detailed computing procedures are presented for the minimum normit χ^2 method plus a numerical example. As an aid to computation, the following tables have been prepared:

(1) Normits (ν) to 5D for $p = 0(.001)1.000$.
(2) Normit weights ($w = z^2/pq$), used in estimating and to 4D for $p = 0(.001)1.000$, where z is the ordinate of the normal curve for a given p and $q = 1 - p$.
(3) Antinormits (p) to 5D for $\nu = 0(.001)2.499$, used to estimate P for a ν estimated from the regression line.

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78[K].—ARTHUR GRAD & HERBERT SOLOMON, "Distribution of quadratic forms and some applications," *Ann. Math. Stat.*, v. 26, 1955, p. 464–477.

Tables I and II of this paper are probabilities $P(Q_k < t)$ for $k = 2$, 3 and $t = .1(.1)1(.5)2(1)5$ to 4S where $Q_k = \sum_{i=1}^{k} a_i x_i^2$ and x_i are normally and independently distributed with zero mean and unit variance, $\sum a_i = 1$, $a_i > 0$. The probabilities for $k = 2$ are given for $a_1 = .5(.1).9$, .95, .99, 1 and for $k = 3$ for $(a_1, a_2) = (1/3, 1/3)$, $(.4, .3)$, $(.4, .4)$, $(.5, .3)$, $(.6, .2)$, $(.5, .4)$, $(.6, .3)$, $(.7, .2)$, $(.8, .1)$. Various approximations are investigated for accuracy. Use of the tables is illustrated with several applications.

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79[K].—G. S. WELSH, "A tabular method of obtaining tetrachoric r with median-cut variables," *Psychometrika*, v. 20, 1955, p. 83–85.

A table is given for obtaining tetrachoric r when it is possible to make cuts at the medians. The computing chart [1] is used to set up a table which can be used if the proportion in the plus-plus cell is known. The table gives r_{tet} to 3D, for proportions to 3D.

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1. L. CHESERE, M. SAFFER, & L. L. THURSTONE, *Computing Diagrams for the Tetrachoric Correlation Coefficient*, University Chicago Bookstore, Chicago, 1933.

80[K].—S. R. BROADBENT, "Quantum hypotheses," *Biometrika*, v. 42, 1955, p. 45–57.

The author considers populations consisting of normally distributed components, particularly if the means of the components differ by a constant amount or quantum, 2δ. The hypothesis that the means of the components are equally spaced is called the quantum hypothesis. The problems treated are the estimation of the quantum which determines the spacing of the modes, the estimate of the scatter within each subdivision, and a test of the quantum hypothesis. Actually the author tests the hypothesis of a rectangular distribution in a subdivision, against the alternative of the quantum hypothesis. This test makes use of the statistic s^2/δ^2 where s^2 is the lumped variance. A table of the critical values of s^2/δ^2 is given for sample sizes of $n = 20(5)100(50)1000$ to 4S for the 5%, 1%, and .1% probability levels.

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81[K].—GARETH HORSNELL, "The determination of single-sample schemes for percentage defectives," *Applied Stat.*, v. 3, 1954, p. 150–158.

In sampling a lot of unknown quality for attributes where each item may be classified as "good" or "defective," it is possible to design a single sampling inspection plan which will accept practically all (say $\alpha = 95\%$) of the lots which are of fraction defective p_1, for example, and which will reject practically all lots (say $1 - \beta = 90\%$) which are of fraction defective $p_2 (p_2 > p_1)$. The single sampling inspection plan for attributes will be given by N, the number of items to be drawn at random for the sample, and K the maximum number of allowable defective items in the sample for which the lot will be accepted. In the case of sampling binomial populations, this problem has been solved by Grubbs [1] for the case $\alpha = .95$ and $\beta = .10$.

The present work by Horsnell covers the case of sampling Poisson populations, i.e., percentage defective less than about 10, and gives some very useful tables for several acceptance and rejection probability levels. In Horsnell's notation, $P(\alpha)$ is the percentage defective for which the chance of acceptance of the lot is α and $P(\beta)$ is the percentage defective for which the chance of acceptance of the lot is β. $[P(\beta) > P(\alpha)$ and $\alpha > \beta.]$ Horsnell gives tables for the Poisson case for $\alpha = .99$ and $\alpha = .95$ and $\beta = .1, .05, .01$ which are tabulated usually to 4S in the form $K = 1(1)20$ vs. $NP(.99)$, $NP(.95)$, $P(.10)/P(.99)$, $P(.05)/P(.99)$, $P(.01)/P(.99)$, $P(.10)/P(.95)$, $P(.05)/P(.95)$, $P(.01)/P(.95)$ and $NP(.50)$.

Thus, one takes the ratio of the objectionable percentage defective $P(\beta)$ to the acceptable percentage defective $P(\alpha)$ for a lot, looks up this ratio in Horsnell's table for the selected α and β and finds the required acceptance number K immediately. The sample size N is then found by dividing the corresponding value of $NP(\alpha)$ to the K found by the acceptable percentage defective $P(\alpha)$.

The tables of the article can be used to obtain sufficient points to plot the operating characteristic of the sampling inspection or the chance that a lot of a

given percentage defective will pass the inspection scheme—and also for finding confidence intervals for the true percentage defective of the lot for an observed number of defectives in a sample of size N drawn at random from the lot.

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1. F. E. GRUBBS, "On designing single sampling inspection plans," *Ann. Math. Stat.*, v. 20, 1942, p. 242–246.

82[K].—S. H. BROOKS, "The estimation of an optimum subsampling number," Amer. Stat. Assn., *Jn.*, v. 50, 1955, p. 398–415.

This paper discusses the use of pilot samples to estimate the optimum subsample size in two-stage sampling. For given ratios between the cost of sampling and population variance in primary and secondary units, and assuming infinite primary and secondary populations, the pilot design of minimum cost such that the average efficiency of the final design relative to the use of the true optimum will be 90%, is given in Table I for cost ratios ≤ 1, 2, 4, 8, 16, 32, 64, 100 and variance ratios $\leq \frac{1}{2}$, 1(1)4, 6, 8, 12, 16, 24, 32, 48, 64. For cost ratios, 1, 2, 4, 8, 16, 32, 64, 100 and subsample size, 1(1)10, 12, 16, 25, 50, 100, Table III gives the upper and lower limits to 1D of the range of values of the variance ratio for which the efficiency will exceed 90%.

These tables and the associated theory should be valuable in connection with other uses of the basic two-factor nested design.

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83[K].—SOLOMON KULLBACK, "The Poisson distribution and its differences up to the 6th order," 1947, vi + 100 p., 26.7 cm. Introduction in typescript; body of table lithoprinted. No copies available but plates are available. Address inquiries to Professor Solomon Kullback, George Washington University, Washington, D. C.

These tables give values of the Poisson frequency function, $(e^{-a}a^x)/x!$ to 7D for $a = 0(.001).01(.01).4$ and to 6D for $a = .5(.1)15(1)100$ which are taken from Molina [1]. In addition the first six ascending differences are given to the same D for the same values of a. They were computed to facilitate the use of the Gram-Charlier type B series which is briefly discussed and illustrated in the introduction.

C. C. C.

1. E. C. MOLINA, *Poisson's Exponential Binomial Limit*, D. van Nostrand Co., New York, 1945.

84[K].—M. A. GIRSHICK, H. RUBIN, & R. SITGREAVES, "Estimates of bounded relative error in particle counting," *Ann. Math. Stat.*, v. 26, 1955, p. 276–285.

This paper gives a method of estimating the parameter λ in a Poisson distribution by gradually expanding the area (or time) observed until a fixed number M

of events occur. If a_M denotes the necessary area, and γ a constant, then confidence limits for λ are $b/[(1 + \gamma)a_M]$ and $b/[(1 - \gamma)a_M]$ with confidence level

$$\alpha = \int_{b/(1+\gamma)}^{b/(1-\gamma)} \frac{x^{M-1}e^{-x}}{(M-1)!}\, dx.$$

A preassigned bound on the percent error in estimating λ is given by 100γ. For fixed M and γ, the value $b = b^* = \dfrac{M(1 - \gamma^2)}{2\gamma} \log \dfrac{1 + \gamma}{1 - \gamma}$ maximizes α, and, for fixed γ, M may be chosen as the least integer so that by using b^*, α is just larger than a preassigned level of confidence.

Table I, using $b = b^*$, gives values of α to 4D for $M = 2(2)40$, $\gamma = .01, .05,$ .10, .20. Most of these tabled α turn out to be small. The largest, for $M = 40$, are .0503, .2479, .4733, .7988 corresponding to the four values of γ. An approximation formula is $\sqrt{4M}\left(\dfrac{\gamma}{2} + \dfrac{7}{48}\gamma^3\right) = z_\alpha$ where z_α is the $100(1 + \alpha)/2$ percentile of the unit normal distribution. A listing for $\alpha = .90, .95, .99$ and $\gamma = .05,$ .10 shows this approximation as good for these cases as a more complicated one which in turn checks with the tabled values for $M = 40$.

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85[K].—LEO KATZ, "Probability of indecomposability of a random mapping function," *Ann. Math. Stat.*, v. 26, 1955, p. 512–517.

Tables are given to 5D for the expression,

$$\frac{(N - 1)!}{(N - i)!} \sum_{M=0}^{N-1-i} N^M/M!,$$

for $i = 0, 1$; $N = 2(1)20(2)40(5)100$. A formula is given for large N. These quantities are the probabilities referred to in the title for $i = 0$ corresponding to a general case and $i = 1$ to the "hollow" case defined in the article. They are computed from values of P given in Molina's tables [1] by the formula,

$$(N - 1)!\, e^N P(N; N - i - 1)/(N - i)^N.$$

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1. E. C. MOLINA, *Poisson's Exponential Binomial Limit*, D. van Nostrand Co., New York, 1945.

86[K].—P. G. MOORE, "The properties of the mean square successive difference in samples from various populations," *Amer. Stat. Assn., Jn.*, v. 50, 1955, p. 434–456.

Three estimators of the parent population variance (σ^2), based on samples of n, are compared: the usual sample variance (s^2) and two definitions based on suc-

cessive differences,

$$\delta^2 = \sum_{i=1}^{n-1} (x_{i+1} - x_i)^2 / (n - 1),$$

$$\eta^2 = \sum_{i=1}^{m} (x_{2i} - x_{2i-1})^2 / 2m,$$

where η^2 is only used for $n = 2m$.

The first four moments of each estimator are derived. The relative efficiencies as estimators of σ^2 (for fixed mean) are tabulated to 2D for $n = 5(5)25$, ∞ and various values of the kurtosis parameter, $\beta_2(= \mu_4/\sigma^4)$.

The bias of the estimators is studied when the population mean shifts; only s^2 is seriously biased. Also the variance of s^2 increases materially, the amount being inversely proportional to β_2.

Since δ^2 has good properties for shifting populations, its distribution is studied. This distribution is best approximated by a Type VI Pearson curve but is close enough to a Type III (χ^2) for most purposes. Four types of parent populations are considered to represent different values of β_2 and the skewness parameter $\beta_1(= \mu_3/\sigma^3)$: Normal ($\beta_1 = 0, \beta_2 = 3$), Rectangular ($\beta_1 = 0, \beta_2 = 1.8$), Double Exponential ($\beta_1 = 0, \beta_2 = 6$), and Type III ($\beta_1 = 1, \beta_2 = 6; \beta_1 = \frac{2}{3}, \beta_2 = 4$). Exact and Type III and Type VI approximate distributions of δ^2/σ^2 are considered for $n = 10(10)50, 75, 100$. The following tabulations are given:

(1) Values of β_1 and β_2 to 4D
(2) Prob $[(\delta^2/\sigma^2) < \xi]$ for selected ξ to 5D for normal parent only
(3) Upper 5% points for δ^2/σ^2 to 2D for normal parent only.

Values of β_1 and β_2 of s^2 and log s^2 are also tabulated for normal (to 2D) and Type III parent populations (to 4D). The log transformation is also advocated for δ^2.

Four examples are given to illustrate the usefulness of δ^2 as an estimator of σ^2.

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87[K].—E. S. Page, "Control chart with warning lines," *Biometrika*, v. 42, 1955, p. 243–257.

First we consider four of Page's five rules. μ is the "ideal" process mean, σ the (known) process standard deviation, N the (regular-interval) sample size, $\bar{x}$ the sample mean.

Rule I. Choose k, n, N. Take samples of size N. Take action if any point ($\bar{x}$) falls outside the action lines ($\mu \pm \beta_1\sigma/\sqrt{N}$) or if any k of the last n points falls outside the warning lines ($\mu \pm \beta_2\sigma/\sqrt{N}, \beta_2 < \beta_1$). Now consider a score x_i assigned to the ith sample as follows: $x_i = -a$ (if the sample point is within the warning lines), $= b$ (if the sample point is between the warning and action lines), $= c$ (if the sample point is outside the action lines) ($a > 0, c > b > 0$). This scoring scheme leads naturally to *Rule II*.

Rule II. Take action after the nth sample if any of the inequalities,
$$\sum_{i=0}^{r} x_{n-1} > h_{r+1}, \quad r = 0, 1, 2, \cdots s,$$ are satisfied, s and h_r suitable constants.

Limiting s only by the number of samples taken since action was last taken but requiring $h_r = h$ for all r, the following two restrictions on *Rule I* are readily obtained (see text and references): (1) $k = 2$ for any n, or (2) $k = n$. We are led to

Rule III. Choose n, N. Take samples of N. Take action if 2 points in any sequence of n fall between the warning and action lines or if any point falls outside the action lines.

Rule IV. Choose n, N. Take samples of N. Take action if n consecutive points fall between the warning and action lines or if any point falls outside the action lines.

For the case of a constant percentage of inspected output, Page suggests that the average run length (the average number of units inspected before the scheme requires action when the parameters remain constant) is a good criterion for selecting an inspection scheme. Page derives and tables average run length to 0D applied to the mean of a normal population, known variance; for *Rule IV* as follows (the mean $\mu' = \mu \pm \lambda \sigma$):

(1) $N = 5$, $n = 3, 4$, $B_1 = 3.00(.125)3.25$, $B_2 = 1.00(.25)2.00$, $\lambda = .00(.02)1.8$
(2) $N = 10$, $n = 3, 4$, $B_1 = 2.875(.125)3.25$, $B_2 = 1.00(.25)2.00$, $\lambda = .00(.02)1.0$
(3) $N = 15$, $n = 3, 4$, $B_1 = 2.75(.125)3.125$, $B_2 = 1.00(.25)2.00$, $\lambda = .00(.02)1.0$
(4) $N = 20$, $n = 3, 4$, $B_1 = 2.75(.125)3.125$, $B_2 = 1.00(.25)2.00$, $\lambda = .00(.02)1.0$.

Finally *Rule V* is: Choose n and N and take action if (i) any point falls outside the action lines, or (ii) n consecutive points fall outside the warning lines, or (iii) two out of any set of n consecutive points fall outside opposite warning lines. For *Rule V* tables show the average run lengths to 0D for $B_1 = 2.875(.125)3.25$ and $B_2 = 1.5(.25)2$ for the two cases: $\mu' = \mu + \lambda \sigma$, $\sigma' = \sigma$ with $\lambda = 0(.2)1$ and $\mu' = \mu$, $\sigma' = k\sigma$ with $k = 1(.25)2.25$.

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88[K].—FELIX RONGE, "Die Verhältnisschätzung (ratio estimate) nach der Methode des 'Veranderungsfaktors' und der 'additiven Veränderungsgrosse,'" *Mitteilungsblatt für Math. Stat.*, v. 6, 1954, p. 221–232.

In this paper the author introduces an additive adjustment method for estimating the total of some population from a sample when information is available from some previous period. Thus $Y_2^* = XN(\bar{y} - \bar{x})$, where Y_2^* is the estimate for the second period, X is the total for the population at the first period, N is the number in the population, $\bar{y}$ is the mean of the sample for the second period, and $\bar{x}$ is the mean of the same sample for the second period.

A comparison of the efficiency of this estimate with the standard ratio estimates is made. This relative efficiency is found to depend upon the correlation, r, between observations made on the two different occasions, the ratio, s, of the standard deviations of individual observations, and the ratio, f, of the two means. For small s and small f or for large s and large f, the additive adjustment method

is found to be more efficient than the ratio estimate. Tables of the relative efficiency to 4D for $r = .6(.1).9, .95, S = .4(.2)2$, and $f = .6(.2)2.4$ are given.

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89[K].—J. E. JACKSON & E. L. ROSS, "Extended tables for use with the 'G' test for means," *Amer. Stat. Assn., Jn.*, v. 50, 1955, p. 416–433.

Define G_1, G_2 as follows: $G_1 = |\bar{X} - u|/\bar{R}$ where $\bar{X}$ is the mean of a sample of size mn, and $\bar{R}$ is the average range of m random non-overlapping subgroups of n; $G_2 = |\bar{X}_1 - \bar{X}_2|/\bar{R}$ where the samples from which $\bar{X}_1, \bar{X}_2$ are obtained are made up of m_1, m_2 random non-overlapping subgroups respectively, of size n each and $\bar{R}$ is the average range of the $m_1 + m_2$ subgroups.

Table I contains percentage points of the distribution of G_1 for significance levels $\alpha = .10, .05$ and $.01$; $n = 2(1)15(2D)$; $m = 1(1)15(2D)$. Table II contains percentage points of the distribution of G_2 for percentage points $\alpha = .10, .05$ and $.01$; $n = 2(1)15(2D)$; m_1 and $m_2 = 1(1)15(2D)$.

Both tables were computed directly from Lord's tables of the percentage points of u, [1] using indicated transformations. For certain values of m, not included in the original tables, corresponding percentage points for u's were obtained using the interpolation formula suggested by Lord.

The indicated tests are of the two-tail variety.

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1. E. LORD, "The use of range in place of standard deviation in the t test," *Biometrika*, v. 34, 1947, p. 41–67.

90[K].—H. F. DINGMAN, "A computing chart for the point biserial correlation coefficient," *Psychometrika*, v. 19, 1954, p. 257–259.

A computing chart is given for quick estimation of a point biserial correlation coefficient when a normally distributed continuous variable is artificially dichotomized at the median. Use is made of [1] and [2] to make a chart which is claimed to have an accuracy to 1D.

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1. L. CHESIRE, M. SAFFIR, & L. L. THURSTONE, *Computing Diagrams for the Tetrachoric Correlation Coefficient*, University of Chicago Bookstore, Chicago, 1933.
2. W. B. MICHAEL, N. C. PERRY, & J. P. GUILFORD, "The estimation of a point biserial coefficient of correlation from a phi coefficient," *Brit. Jn. Psych.*, Stat. Sec. 1952, v. 5, p. 139–150.

91[K].—R. F. TATE, "The theory of correlation between two continuous variables when one is dichotomized," *Biometrika*, v. 42, 1955, p. 205–216.

Karl Pearson's biserial correlation coefficient r^* is shown to be consistent and asymptotically normal as an estimate of ρ, the bivariate normal correlation. If ω is the (standardized) point of dichotomy, and $p(\omega) = \int_\omega^\infty (2\pi)^{-\frac{1}{2}} e^{-t^2/2} dt$, the dis-

tribution of r^* depends upon ρ and p. Soper [1] obtained the asymptotic variance,

$$A V(r^*) = \frac{1}{n} \left\{ \rho^4 + \rho^2 \left[\frac{pq\omega^2}{\lambda^2} + \frac{(2p - 1)\omega}{\lambda} - 5/2 \right] + \frac{pq}{\lambda^2} \right\},$$

which is a function of ρ and p only, λ being the normal density with argument ω.

Table 2 gives $\{nA V(r^*)\}^{\frac{1}{2}}$, the square root of the expression in curly brackets above, to 3D (3 and 4S) for $\pm \rho = 0(.10)1$ and p or $(1 - p) = .05(.05).5$.

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1. H. E. SOPER, "On the probable error for the biserial expression for the correlation coefficient," *Biometrika*, v. 10, 1913, p. 384–390.

92[K].—J. L. HODGES, JR., "A bivariate sign test," *Ann. Math. Stat.*, v. 26, 1955, p. 523–527.

Consider n pairs of bivariate vectors (x_i, y_i) and (x_i', y_i'), $i = 1, \cdots, n$. The $2n$ vectors of this set are assumed to be statistically independent. The null hypothesis is that the bivariate distribution for (x_i, y_i) is identical with that of (x_i', y_i') for $i = 1, \cdots, n$. The alternative hypothesis of interest is that the probability distribution for (x_i', y_i') has been shifted relative to that for (x_i, y_i) and that this unknown shift direction is roughly the same for all i. For given shift direction, the problem reduces to the sign test by projecting the vectors of differences $(x_i - x_i', y_i - y_i')$ onto the given direction and counting the number S of these vectors which have a specified one of the two possible direction senses. Consider all possible shift directions and let M be the maximum of the corresponding S values. The null hypothesis is rejected if M is too large. A procedure is given for determining M for all possible cases. The quantity $K = n - M$ is used for the derivations and tabulation. For $k < n/3$,

$$\Pr (K \leq k) = (n - 2k) \binom{n}{k} / 2^{n-1}.$$

The table in this article lists values of $\Pr (K \leq k)$ to 5D for $n = 1(1)30$ and $k < n/3$.

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93[K].—D. R. COX & A. STUART, "Some quick sign tests for trend in location and dispersion," *Biometrika*, v. 42, 1955, p. 80–95.

In a linear regression model with $y_i = \alpha + \Delta i + \epsilon_i$, $(i = 1, 2, \cdots, N)$, the ϵ_i are independent standardized normal variates. The null hypothesis, $\Delta = 0$, is to be tested against alternatives $\Delta > 0$, using a (quick) test based on comparisons of *independent* pairs of the observations. It is shown that the best weighted sign test is given by $S_1 = \sum_{K=1}^{N/2} (N - 2K + 1)h_{K, N-K+1}$, and the best unweighted sign

test is given by $S_3 = \sum_{K=1}^{N/3} h_K, \frac{2N}{3} + K$ where, for $i < j$, $h_{ij} = 1$ if $y_i > y_j$ and $h_{ij} = 0$ otherwise. Since the asymptotic efficiency of S_3 relative to S_1 is 96%, the authors recommend use of S_3. It is also established that the asymptotic relative efficiency of S_3 compared to the best parametric test based on b, the sample regression coefficient, is 83%.

Table 3 gives the exact power of S_3 to 3D for sample sizes $N = 15(15)135$, significance level the largest value $\leq .05$, and alternatives given by $p = .50, .49, .45(.05).05$, where $p = \Phi\left(\frac{-\sqrt{2}N\Delta}{3}\right)$, the unit normal c.d.f. The values of p are identified as the true significance level α, and the rows $a, b, \cdots, j$, give powers for the various Δ's used. Table 4 gives the exact power of the b test to the same accuracy for the same alternatives. Table 3 also shows, in parentheses, the values of the corresponding standardized regression coefficient, Δ, to 4D. The smallest ratio of powers tabulated is 50.3%, for $N = 30$, p $= .25$ and $\Delta = .0477$.

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94[K].—EVELYN FIX & J. L. HODGES, JR., "Significance probabilities of the Wilcoxon test," *Ann. Math. Stat.*, v. 26, 1955, p. 301–312.

The problem considered is that of obtaining significance probabilities for the Wilcoxon unpaired two-sample test [1]. Let $m \leq n$ be positive integers while $R_1 < R_2 < \cdots < R_m$ represent a random sample of size m drawn without replacement from the first $m + n$ positive integers. Let $S_i = R_i - i$ and $U = S_1 + \cdots + S_m$ while $\pi(u, m, n) = \Pr(U \leq u)$. The value of $\pi(u, m, n)$ can be expressed in the form $A(u, m, n)\Big/\binom{m+n}{n}$, where

$$A(u, m, n) = \sum (-1)^k A_k[u - kn - \tfrac{1}{2}k(k+1), m - k]$$

and the summation is over all non-negative values of the integer k for which $u - kn - \frac{1}{2}k(k+1)$ is non-negative. Table I contains exact values of $A_0(u, m)$ for $m \leq 12$ and $u \leq 100$ while Table II contains exact values of $A_2(u, m)$ for $m \leq 11$ and $u \leq 75$. In general, $A_k(u, m)$ for any k can be found from

$$A_k(u, m) = \sum A_{k-1}(u - rk, m)$$

where summation is over all non-negative integers r such that $u - rk$ is non-negative. To extend the range of validity past $m \leq 12$ and $u \leq 100$, the Edgeworth approximation to $\pi(u, m, n)$ is given to terms of order $1/m^2$. This approximation appears to be accurate to about 4D for $m \geq 12$.

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1. F. WILCOXON, "Individual comparisons by ranking methods," *Biometrica*, v. 1, 1945, p. 80–83.

95[K].—DES RAJ, "Relative efficiency of gauging and exact measurement in estimating the proportion of a population between given limits," *Sankhya*, v. 15, 1955, p. 191–196.

Suppose that we want to estimate the proportion, P, of a population falling in the interval (a, b). The author compares the large sample relative efficiency of two methods for estimating this proportion; one method consisting of measuring each item in the sample, the other method consisting of using go-not go gauges set at the values a and b. Tables for the proportion P and the efficiency are given for the Type III, Cauchy, and normal distributions, for the case where a and b are symmetrically located about the mean of the distribution, and for the one-sided case where $b = \infty$. In the normal case efficiencies are tabulated to 3D for those values $a = \mu - \lambda\sigma$ for which $\lambda = 0(.05)3.00$.

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96[K].—R. E. GREENWOOD, "Coupon collector's test for random digits," *MTAC*, v. 9, 1955, p. 1–5.

In an ordered set of digits beginning with a specified position one may determine the length of the sequence of digits required in order to have the complete set $0, 1, \cdots, 9$ appear in it. The author proposes that the comparison of the observed lengths of non-overlapping sequences with the theoretical distribution of such lengths, on the assumption that the probability that any one of the ten values be assumed by any digit is $1/10$, be used as a test of randomness in the ordered set. If p_n is the probability that a sequence have the length n, $p_n = \dfrac{\Delta^9 0^{n-1}}{10^{n-1}}$, $n = 10, 11, \cdots$. In order to tabulate values of p_n, the author extended the table of differences of zero of Fisher and Yates [1]. His table is claimed to give exact values of p_n for $n = 10(1)35$ and values to 20D for $n = 36(1)75$. A check through $n = 26$ disclosed one error, apparently typographical. For $n = 13$, p_n should read 0.0080 8315 2 instead of 0.0080 9315 2. This correction was noted with an obvious misprint, *MTAC*, v. 9, p. 224. This coupon collector's test is illustrated on the 2035D approximation to π and the 2500S approximation to e. A χ^2 test, using grouped frequencies, gave no indication of non-randomness in these two interesting ordered sets.

C. C. C.

1. R. A. FISHER & F. YATES, *Statistical Tables for Biological, Agricultural and Medical Research*, 3rd edition, London, 1948, Table XXII.

97[K].—G. W. THOMPSON, "Bounds for the ratio of range to standard deviation," *Biometrika*, v. 42, 1955, p. 268–269.

This table gives upper and lower bounds (distribution free) for the ratio of the range w to standard deviation estimate s, both from the same sample of size n. It can be shown that the upper bound of w/s is $\sqrt{2(n-1)}$ and the lower bound of w/s is $2\sqrt{n - 1/n}$ for n even and $2\sqrt{n/n + 1}$ for n odd.

A table is given for $n = 3(1)20(10)100, 150, 200, 500, 1000$, to 3D. For samples

of size 3 from a normal population, a table of upper and lower percentage points of w/s is given for percentages 0, 0.5, 1.0, 2.5, 5.0, and 10.0, to 5D.

These can be used for routine checks of computation of s.

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98[K].—T. J. TERPSTRA, "A non-parametric test for the problem of k samples," K. Ned. Akad. v. Wetensch., *Proc.*, v. 57, s. A., 1954, p. 505–512.

Let $X_1, \cdots, X_k$ be independent random variables and let H_0 be the hypothesis that all the X_i have the same continuous distribution function. Let $x_{i,h}(h = 1, \cdots, n)$ be a set of n_i observations on $X_i(i = 1, \cdots, k)$. For $i \neq j$, let $U_{ij} + \frac{1}{2}n_i n_j$ equal the number of times that $x_{i,h} > x_{j,m}(h \leq n_i, m \leq nj)$, while $U_{ii} = 0$. Finally,

$$Q = 6 \sum_{i,j} \frac{U^2_{ij}}{n_i n_j} - \frac{12}{n+1} \sum_i \frac{1}{n_i} \left(\sum_j U_{ij} \right)^2,$$

where $n = \sum_i n_i$. Let $n_i = a_i m + 0(\sqrt{m})$, where a_i and m are positive integers, a_i being independent of m. It is shown that, under H_0 and for large m, the statistic Q has approximately a χ^2-distribution with $k(k-1)/2$ d.f. Further, in Table 1 the exact distribution of Q to 4D is given for $k = 3$, $n_1 \leq n_2 \leq n_3 \leq 3$, while Table 2 gives the corresponding upper and lower .10, .05, and .01 significance levels also to 4D. In Table 3, for $n_1 = n_2 = n_3 = 3$, the exact distribution of Q is compared with the limiting χ^2-distribution, indicating that for $n_i > 3$ the χ^2-approximation is quite satisfactory.

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99[K].—M. ZIA UD-DIN, "Expression of the K-statistics k_9 and k_{10} in terms of power sums and sample moments," *Ann. Math. Stat.*, v. 25, 1954, p. 800–803.

To the k-statistics of Fisher expressed in terms of the power sums of the observations in a sample of n which had been given through weight 8 by Dressel [1], the author adds k_9 and k_{10}.

C. C. C.

1. PAUL L. DRESSEL, "Statistical Semi-invariants and their estimates with particular emphasis on their relation to algebraic invariants," *Ann. Math. Stat.*, v. 11, 1940, p. 33–57.

100[K, W].—D. R. Cox, "A table for predicting the production from a group under the care of one operative," Roy. Stat. Soc., *Jn.*, ser. B, v. 16, 1954, p. 285–287.

For the case of N machines under the care of a single operator, assuming random occurrence of stoppages and an exponential distribution of clearing times, the expected fraction of the operator's time utilized is tabulated to 3D for $x = .005(.001).01(.01).12$ and $Nx = .1(.1)1.9$ (also $Nx = .85(0.1)1.15$ for $x \leq 0.04$), where x is the product of the mean clearing time and mean ratio of

· occurrence of stops. This table essentially extends Tables 3 and 4 of Benson and Cox [1] to larger N. A simple relation enables one to compute the expected production rate per machine.

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1. F. BENSON & D. R. COX, "The productivity of machine requiring attention at random intervals," Roy. Stat. Soc., *Jn.*, ser B, v. 13, 1951, p. 65–82.

101[L].—NATIONAL PHYSICAL LABORATORY, *Tables of Weber Parabolic Cylinder Functions*. Computed by Scientific Computing Service Limited, Mathematical Introduction by J. C. P. Miller, Editor. H. M. Stationery Office, London, 1955, 233 p., 28 cm. Available from British Information Services, 30 Rockefeller Plaza, New York 20, New York.   Price $11.34.

This book offers much more than the title seems to promise. More than one third of the book is taken up by a mathematical introduction containing all the information needed, by a computer, on Weber's differential equation and its various solutions. The remaining part of the book contains the numerical tables.

Weber's differential equation is written in the alternative forms

(1)
$$\frac{d^2y}{dx^2} - (\tfrac{1}{4}x^2 + a)y = 0$$

(2)
$$\frac{d^2y}{dx^2} + (\tfrac{1}{4}x^2 - a)y = 0.$$

In terms of the parabolic cylinder function $D_a(z)$, and of the function

$$E(a, x) = \left[\frac{4ie^{\pi a}\Gamma(\tfrac{1}{2} + ia)}{\Gamma(\tfrac{1}{2} - ia)} \right]^{\tfrac{1}{2}} D_{-ia-\tfrac{1}{2}}(xe^{-i\pi/4})$$

relevant solutions of (1), for real a, may be defined as

$$U(a, x) = D_{-a-\tfrac{1}{2}}(x), \qquad \bar{U}(a, x) = \tan \pi a\, U(a, x) + \sec \pi a\, U(a, -x)$$
$$V(a, x) = \bar{U}(a, x)/\Gamma(\tfrac{1}{2} - a), \quad \bar{V}(a, x) = U(a, x)/\Gamma(\tfrac{1}{2} - a),$$

and relevant solutions of (2), for real a, as

$$W(a, x) = k^{\tfrac{1}{2}} \operatorname{Re} E(a, x), \quad W(a, -x) = k^{-\tfrac{1}{2}} \operatorname{Im} E(a, x),$$
where
$$k = \sqrt{1 + e^{2\pi a}} - e^{\pi a}.$$

These solutions have been selected as being most suitable for numerical tabulation.

The mathematical introduction by J. C. P. Miller contains power series expansions; asymptotic expansions (i) for large $|x|$, (ii) for large $|a|$, (iii) for $|a|$ and $|x|$ both large; integral representations; relations with Bessel functions and other special functions; a very extensive collection of formulas; a description of

the tables together with an account of their preparation and instructions for interpolation; and a bibliography of previous tables and researches connected with parabolic cylinder functions. Both differential equations, (1) and (2), are included in this discussion. In the course of the work certain slowly varying auxiliary functions are introduced for computational purposes.

The numerical part contains tables relating to solutions of (2). Certain numerical tables of solutions of (1), especially for integer values of $2a$, are available but on the whole a systematical tabulation of solutions of (1) has not as yet been undertaken.

Almost all computations were carried out by Scientific Computing Service, Ltd., and were supervised by J. C. P. Miller who was supported in this task by the late L. J. Comrie, and by C. W. Jones. The Mathematics Division of the National Physical Laboratory cooperated in the production of the volume.

Table I gives $W(a, x)$, $W(a, -x)$ and reduced derivatives for $a = -10(1)10$, $x = 0(.1)10$. W is generally given to 8S, and as many reduced derivatives are given as can be used effectively in interpolation in the x direction. The a-interval is too large to permit interpolation in the a direction.

Table II gives auxiliary functions in the non-oscillatory region. $\log W(a, \pm x)$ and the first derivatives of these functions are given to 8S with second and fourth (or modified second) central differences for $a = 1(1)10$, $x = 0(.1)2\sqrt{a}$.

Table III gives auxiliary functions in the oscillatory region: these are defined by the equations

$$E(a, x) = Fe^{ix}, \quad dE(a, x)/dx = -Ge^{i\psi},$$

F, G, χ, ψ real. 8S values of F, G, χ, ψ, with differences as in Table II, are given for $a = -10(1)2$, $x = 0(.1)10$ and $a = 3(1)10$, $x = 2\sqrt{a}(.1)10$.

Tables IV to VI give auxiliary constants: $e^{\pm \pi a}$, $k^{\pm \frac{1}{2}}$, $1 - k^{\pm \frac{1}{2}}$ for $a = 0(.1)10$; $e^{-\pi a}$ for $a = 0(.001).1$; $W(a, 0)$ and $W'(a, 0)$ for $a = 0(.02)1(.1)5(.2)20$; real and imaginary parts of $\log \Gamma(n/4 + ia/2)$ for $n = 0(1)3$, $a = 0(.1)5(.2)20$.

This book is a model of presentation. It contains all the material needed by the computer, and its arrangement is admirable. The printing is excellent.

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102[L].—M. Schuler & H. Gebelein, (a) *Acht- und neunstellige Tabellen zu den elliptischen Funktionen* (*Eight and Nine Place Tables of Elliptical Functions*), Springer, Berlin, 1955, xxiv + 296 p., 26.7 cm. Price DM 58, (b) *Fünfstellige Tabellen zu den elliptischen Funktionen* (*Five Place Tables of Elliptical Functions*), same publishers, 1955, xi + 114 p., 23.8 cm. Price DM 29.60.

Both these volumes contain tables of Jacobian elliptic functions and theta functions, or related auxiliary functions, on a new system. The novelty lies chiefly in the arguments. As modular argument, instead of the usual modular angle (here called Θ) or of $k^2 = \sin^2 \Theta$, use is made of Jacobi's nome q (or of q^2 or q^4). As the other argument, instead of the usual argument u of the Jacobian elliptic functions or of the argument $x = \pi u/2K$ often used for the theta functions, the authors

make use of $z = \cos 2x = \cos (\pi u/K)$. The aim of the new system is ease of bivariate interpolation. This aim has been notably achieved, and the tables promise to be useful when interpolation in both arguments is unavoidable; such interpolation is sometimes avoidable when Θ or k^2 is a natural argument in the problem to which elliptic functions are being applied.

The authors consider that the advantages of their system would be lost if ordinary bivariate interpolation were used. Their bivariate tables are meant to be interpolated first in one variable and then in the other, either way round according to circumstances. Consequently these tables are given twice over, first arranged with q (or q^3 or q^4) as top argument and z as left argument, with differences in z, and then vice versa.

In the detailed descriptions which follow, the first-mentioned argument in bivariate tables is the top argument, and differences relate to the second-mentioned (left) argument. All logarithms are to the base 10. The functions G, H, $\bar{G}$, $\bar{H}$ used by the authors are related to ordinary theta functions by the equations

$$\vartheta_1(x) = 2q^{\frac14} \sin x \bar{G} \quad \vartheta_3(x) = \bar{H}$$

$$\bar{G} = 1 - q^2 G \qquad \bar{H} = 1 + qH.$$

The detailed contents of (a) are:

Table I. $G(q^4, z)$ to 9D with Δ^2 for $q^4 = .001(.001).1$, $z = -1(.05) + 1$. Also q to 7D and Θ to $0'.0001$, both without differences, for the same q^4.

Table II. $G(q^4, z)$ to 9D with Δ for $z = -1(.05) + 1$, $q^4 = 0(.001).1$. Also Θ and q again.

Table III. $H(q^3, z)$ to 9D with Δ^2 for $q^3 = .002(.002).176$, $z = -1(.05) + 1$. Also q to 7D and Θ to $0'.0001$, both without differences for the same q^3.

Table IV. $H(q^2, z)$ to 9D with Δ for $z = -1(.05) + 1$, $q^3 = 0(.002).176$. Also Θ and 2 again.

Table V. $\log (\operatorname{sn} u/\sin x)$, $\log (\operatorname{cn} u/\cos x)$, $\log dn\, u$ to 8D with Δ for $q = .01(.01).55$, $z = -1(.05) + 1$. Also $-\log \cos \Theta$ to 8D, Θ to $0'.01$, and K, K/E to 8D, all without differences, for the same q.

Table VI. $\log (\operatorname{sn} u/\sin x)$, $\log (\operatorname{cn} u/\cos x)$, $\log dn\, u$ to 8D with Δ for $z = -1(.05) + 1$, $q = 0(.01).55$. Also $-\log \cos \Theta$ to 8D and Θ to $0'.01$, both without differences, and K, K/E to 8D with Δ, all for the same q.

Table VII. $1/(1 - q)$, K, K/E, all to 8D with Δ for $-\log k' = -\log \cos \Theta = 0(.005)3$. The values of K and E for $-\log k' > 0.5$ were obtained from a table by E. L. Kaplan [1].

The detailed contents of (b) are:

Table I. $\log (\operatorname{sn} u/\sin x)$, $\log (cn\, u/\cos x)$, $\log dn\, u$ to 5D with Δ or Δ^2 (the second difference being modified when italicized) for $q = .01(.01).5$, $z = -1(.1) + 1$. Also Θ to $0'.01$ without differences for the same q.

Table II. $\log (\operatorname{sn} u/\sin x)$, $\log (\operatorname{cn} u/\cos x)$, $\log dn\, u$ to 5D with Δ or Δ^2 for $z = -1(.1) + 1$, $q = 0(.01).5$. Also Θ to $0'.01$ and $-\log \cos \Theta$, K, K/E to 5D, all without differences for the same q.

Table III. $\bar{G}(q, z)$, $\bar{H}(q, z)$ to 5D with Δ or Δ^2 for $q = .01(.01).5$, $z = -1(.1) + 1$. Also Θ to $0'.01$ without differences for the same q.

Table IV. $\bar{G}(q, z)$, $\bar{H}(q, z)$ to 5D with Δ^2 for $z = -1(.1) + 1$, $q = 0(.01).5$.

Table V. $1/(1 - q)$, K, K/E to 5D with Δ, and Θ to $0'.01$ without differences, all for $-\log k' = -\log \cos \Theta = 0(.01)2.5$.

Table VI. Coefficients of second differences for Everett interpolation of both function and first derivative, both to 4D with Δ for fraction of interval = $0(.01)1$.

Both volumes have explanatory text in German and English with worked examples and illustrative diagrams.

A. F.

1. E. L. KAPLAN, "Auxiliary table of complete elliptic integrals," *J. Math. Phys.*, v. 25, 1946, p. 26–36, RMT 311, *MTAC* v. 2, 1946–7, p. 127–128.

103[L].—F. G. TRICOMI, "Valori numerici di funzioni ortogonali di Laguerre," *Atti. R. Acc. Sc. Torino*, v. 90, 1955–1956, p. 1–8.

This is a revision and extension of the author's earlier table [1] which was reproduced in Jahnke-Emde [2]. It gives $l_n(x) = e^{-\frac{1}{2}x}L_n(x) = e^{-\frac{1}{2}x} \sum_{m=0}^{n} \binom{n}{m} \frac{(-x)^m}{m!}$ to 6S for $n = 0(1)10$, $x = 0.1(.1)1(.25)6(1)14(2)34$. The range was chosen so as to include all zeros of all the functions. The present table was prepared on a desk calculator. The computations were carried out to 8S using the recurrence relation, $(n + 1)l_{n+1} = (2n + 1 - x)l_n - nl_{n-1}$. This method is known to be dangerous due to the building up of rounding-off errors, so that thorough checking is essential.

Consider $\mathcal{L}_n(x) = n!L_n(x) = \sum_{m=0}^{n} \binom{n}{m}^2 (n - m)!(-x)^m$. This has integral values for integral arguments, and these can be calculated exactly by the use of the recurrence relation

$$\mathcal{L}_{n+1} = (2n + 1 - x)\mathcal{L}_n - n^2\mathcal{L}_{n-1}.$$

From these the corresponding values of $l_n(x)$ were obtained by multiplication by $e^{-\frac{1}{2}x}/n!$. The other values tabulated were obtained by use of an addition formula [3]

$$l_n(x + h) = [l_n(h) - l_{n-1}(h)]l_0(x) + [l_{n-1}(h) - l_{n-2}(h)]l_1(x) + \cdots$$
$$+ [l_1(h) - l_0(h)]l_{n-1}(x) + l_0(h)l_n(x),$$

and a bisection formula [3]

$$l_n(\tfrac{1}{2}x) = e^{\frac{1}{4}x}2^{-n}\left[l_0(x) + \binom{n}{1} l_1(x) + \binom{n}{2} l_2(x) + \cdots + \binom{n}{n} l_n(x)\right].$$

The table was compared with that of L. J. Slater [4], where the arguments coincided. After multiplication by the appropriate exponential there was agreement, up to 1 or 2 units in the last place, except for 6 values. Dr. Slater informed Professor Tricomi that his values were correct; they are

$$L_6 (5) = -2.090\ 278, \quad L_7 (5) = 0.325\ 397,$$
$$L_8 (3.5) = -1.252\ 437, \quad L_9 (3.5) = -0.584\ 567,$$
$$L_{10}(3.5) = 0.221\ 114, \quad L_{10}(3) = -0.700\ 022.$$

Dr. Slater informed the reviewer that errors were possible whenever a different routine was used for the calculation when scaling difficulties arose: this routine came into action at $x > 4.1$, 3.4, 2.8, 2.4, 2.1, 1.9 for $n = 5$, 6, 7, 8, 9, 10 respectively.

<div align="right">J. T.</div>

1. F. G. TRICOMI, "Generalizzazione di una formula asintotica sui polinomi di Laguerre e sue applicazioni," Accad. delle Scienze di Torino, Cl. d. sci. fis., mat., e nat., *Atti*, v. 76, 1941, p. 281–316.
2. E. JAHNKE & F. EMDE, *Tafeln höherer Funktionen*, ed. 4, Teubner, Leipzig, 1948, p. 32–33.
3. F. G. TRICOMI, *Vorlesungen über Orthogonalreihen*, Springer, Berlin, 1955, p. 218.
4. LUCY J. SLATER, "A short table of the Laguerre polynomials," Inst. Elec. Engineers, Monograph 136, 1955. To be republished in Inst. Elec. Engineers, *Proc.*, Part C. [See *MTAC*, v. 10, 1956, p. 174–175, which contains further bibliographical material.]

104[L].—NBS Applied Mathematics Series, No. 37, *Tables of Functions and of Zeros of Functions*, U. S. Gov. Printing Office, Washington, D. C., 1954, ix + 211 p., 30 cm. Price $2.25.

This is the first of a series of volumes, to be called *Collected Short Tables of the National Bureau of Standards Computation Laboratory*, and to contain some tables previously printed elsewhere as well as certain tables, hitherto unpublished, which have been found useful in the operation of the Computation Laboratory.

In the present volume 18 tables are given. Of these, 15 have either been published or reviewed in *MTAC*; the remaining three have not been previously published. All tables are reproduced by a photo-offset process, and in the case of published tables the explanatory text is also reproduced. In addition, there is an Introduction by A. N. Lowan to all 18 tables, giving, in some cases, additional references.

1. On the function $H(m, a, x)$. See *MTAC* 1, 1944, RMT **135**, p. 156.

2. Table of the integrals $\int_0^x J_0(t)dt$ and $\int_0^x Y_0(t)dt$. See *MTAC* 1, 1944, RMT **133**, p. 154.

3. Table of $Ji_0(x)$ and related functions. See *MTAC* 1, 1944, RMT **134**, p. 155.

4. Table of $f_n(x)$. See *MTAC* 1, 1945, RMT **181**, p. 363–364.

5. The function $E_n(x)$. See *MTAC* 2, 1947, RMT **392**, p. 272.

6. Table of the Struve functions. See *MTAC* 2, 1947, RMT **387**, p. 268–269.

7. Table of Fourier coefficients. See *MTAC* 1, 1944, RMT **157**, p. 192–193.

8. Table of sines and cosines. Gives $\sin x$ and $\cos x$ for $x = 100(1)1000$ to 8D (guaranteed to within a unit in the last place). This supplements the tables in NBS AMS 43.

9. Radix tables for finding logarithms to 25 decimal places, by Herbert E. Salzer. Gives 28D values of $\log_e x$ for $x = 2(1)10$, and $\log_e (1 \pm 10^{-n})$ for $n = 1(1)14$, together with instructions for using this brief table for the computation of logarithms.

10. Table of $x^n/n!$ for $x = .01(.01)1.99$ to 13D for n taken up to the point where $x^n/n!$ vanishes to 13D, and also for $x = 1(1)10$, $n = 1(1)40$ to 8S. (Both tables are guaranteed to within one unit in the last place.)

11. Zeros of Legendre polynomials. See *MTAC* 1, 1943, RMT **92**, p. 51–52.

12. Zeros and weight factors of Laguerre polynomials. See *MTAC* 4, 1950, RMT **737**, p. 86.

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105[L, M, S].—A. ASHOUR & A. SABRI, "Tabulation of the function $\psi(\theta)$ $= \sum_{n=1}^{\infty} \dfrac{\sin n\theta}{n^2}$," *MTAC*, v. 10, 1956, p. 57–65.

This article contains a table of the function $\sum_{n=1}^{\infty} \sin n\theta/n^2$, $\theta = 0°(10')180°$, 6D, with first differences. It is prefaced by an interesting introduction which describes the functional properties and the method of computation. This series is of particular interest from the point of view of numerical analysis. The reader might contemplate the difficulties he must face in evaluating this series to the accuracy given in the table by straightforward summation on the most modern high-speed computer. The authors avoided these difficulties by working with the integral representation

$$\psi(\theta) \equiv \psi(0, \theta) = - \int_0^\theta \log (2 \sin \tfrac{1}{2}t)dt$$

to first obtain the values from $\theta = \tfrac{1}{2}\pi$ to π by numerical quadrature. Thus the logarithmic singularity at $t = 0$ was avoided by making use of the relation $\psi(0, \tfrac{1}{2}\pi + \theta) = - \psi(\tfrac{1}{2}\pi + \theta, \pi)$. In the range $t = \tfrac{1}{2}\pi$ to $t = \pi$ the integrand is continuous and, working with an interval of 10', the trapezoidal rule, corrected to take account of the first derivative, gave satisfactory results. The values of the integrand $\log_{10} \sin \tfrac{1}{2}t$ and its derivative were available from existing tables. The values from $\theta = 0$ to $\theta = \tfrac{1}{2}\pi$ were obtained with the functional equation $\psi(\theta) = \psi(\pi - \theta) + \tfrac{1}{2}\psi(2\theta)$. The function is tabulated for $\theta = 0°(10')180°$ to six places, since they were all readily obtainable as a result of the method of computation. Although the table is not linearly interpolable near the beginning of the table, first differences are given throughout. Since

$$\psi(\pi - \theta) = \theta \log_e 2 - \sum_{n=1}^{\infty} \frac{(-)^{n-1}}{(2n)!} B_{2n}(2^{2n} - 1) \frac{\theta^{2n+1}}{2n(2n + 1)},$$

we have $\psi(\theta) \sim (\pi - \theta) \log_e 2$ near $\theta = \pi$ and so the tables could have been at intervals of 20' rather than 10' from 150°–180° and still satisfy the criterion for linear interpolation. As a check on the tables the sixth differences were computed for successive groups of seven entries using the last value of one group as the first value of the next. The results indicate that the values are probably correct to within a unit in the last place given. The given first differences were checked by summation and it was found that the first difference for $\theta = 179° 40'$ should be -2017, not -2016.

The tables could have been reduced in size considerably if standard practice were followed. For example in the neighborhood $\theta = 0°$ the authors give the representation

$$\psi(\theta) = -\theta \log|\theta| + \theta + \sum_{n=1}^{\infty} \frac{(-)^{n-1}}{(2n)!} B_{2n} \frac{\theta^{2n+1}}{2n(2n+1)}.$$

The presence of the logarithmic singularity makes interpolation unfeasible near $\theta = 0°$. However, if the auxiliary function $\bar{\psi} = \psi(\theta) + \theta \log|\theta|$ were given we would obtain (using the tabular entries)

θ	$\bar{\psi}$
0°	.000000
1°	.017453
2°	.034908
3°	.052362
4°	.069818
5°	.087276

The resulting table is linear and the 31 entries given in the table could be replaced by the six entries above. The values of $\psi(\theta)$ could be given to make a total of 12 entries instead of 31. The values of $\theta \log|\theta|$ can be obtained with the aid of a table of logarithms.

For the remainder of the table if a criterion for interpolation with modified second differences using Everett's formula had been adopted the following intervals would have been sufficient

θ	Interval	No. of arguments
5°–10°	30′	10
10°–30°	1°	20
30°–90°	2°	30
90°–180°	5°	18

Thus the total number of arguments necessary to provide the same information would be 84, with 90 functional values and their corresponding modified second differences. The resulting table would require less than one page rather than the six pages as given.

It may be argued by some that the ideas mentioned above are given from the point of view of the table maker and table users would prefer the table in its present form, long as it is. However, the appearance of this table and one or two others in earlier issues indicates a change in editorial policy of *MTAC*. Previously only short tables of unusual interest were given in *MTAC*. It is the opinion of the reviewer that this policy should be continued and consequently every device leading to economization in the size of tables appearing in *MTAC* should be employed. This will avoid the possibility that there be a slack in articles on numerical analysis, bibliographic information, reviews, etc. It is clear that the editors are trying to fill a need in providing a place where tables can be published—a need which was indicated at the Conference on Mathematical Tables held at MIT in September, 1954 (see *MTAC* 9, 1955, p. 42–44). However, it is question-

able whether *MTAC* should be used for this purpose—it would seem that tables might be better published in a supplementary issue or perhaps a new journal be started for this purpose.

This function had been tabulated for much coarser interval by T. Clausen [1] and Clausen's table is reprinted in [2].

<div align="right">A. ABRAMOWITZ</div>

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1. T. CLAUSEN, "Über die Zerlegung reeller gebrochener Funktionen," *J. f. d. reine u. angew. Math.* (Crelle), v. 8, 1832, p. 298–300.
2. F. W. NEWMAN, *The Higher Trigonometry*, McMillan and Bowes, Cambridge, 1892.

106[L].—J. R. PHILIP, "Numerical solution of equations of the diffusion type with diffusivity concentration-dependent," Commonwealth Scientific and Industrial Research Organization, Faraday Soc., *Trans.*, No. 391, v. 51, part 7, 1955, p. 885–892.

This table contains a table of the function $A(x)$ for $x = 0(.2)1(.5)4(1)10$, 3D. The function $A(x) = $ ierfc $(x/\text{erfc } x)$. The calculation of the value of the function was based on an alternate exponential of the function and the use of tables in [1]. An asymptotic expansion for $A(x)$ is included.

Then a tabulated function is introduced in connection with an iterative method for the solution of a diffusion equation of the type

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(D \frac{\partial \theta}{\partial x}\right),$$

where D is a function of θ.

<div align="right">C. B. T.</div>

1. L. J. COMRIE, *Chambers's Six-Figure Mathematical Tables*, v. 2, Chambers, London, 1959, p. 518.

107[M].—A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER, & F. G. TRICOMI, *Tables of Integral Transforms*, McGraw-Hill Book Co., Inc., New York, Toronto, London, v. 1, 1954, xx + 391 p. Price $7.50; v. 2, 1954, xvi + 451 p. Price $8.00.

This work is unique in more than one respect. It is the first extensive collection of functional relations involving integrals of all those functions which may be called hypergeometric in the widest possible sense. There are integrals containing Gauss' series, its familiar special cases (such as Legendre functions, Jacobi polynomials and the complete elliptic integrals), its limiting cases (such as Bessel and Whittaker functions), and many of its generalizations (among them Meijer's *G*-function and Appell's and Horn's hypergeometric functions of two variables). There are also integrals involving Theta functions. There are no integrals with Mathieu or Spheroidal wave functions, and no systematic attempt is made to list integrals involving elementary functions only. (For the latter the authors rightly refer to the recent excellent tables by W. Meyer zur Capellen and by W. Gröbner and N. Hofreiter.)

Unique, too, is the organization of the tables. If possible, the integrals are listed as integral transforms. This means that they are written in the form

$$g(y) = \int_a^b K(x, y)f(x)dx,$$

where the kernel K and the limits a and b are standardized. The following transforms have been considered (in order to adhere to conventional notations, other letters than x and y are occasionally used):

Name of transform	$K(x, y)$	a	b	No. of pages
Fourier cosine	$\cos (xy)$	0	∞	56
Fourier sine	$\sin (xy)$	0	∞	54
Exponential Fourier	e^{-ixy}	$-\infty$	∞	8
Laplace	e^{-xy}	0	∞	98
Inverse Laplace	e^{xy}	$c - i\infty$	$c \quad i\infty$	75
Mellin	x^{y-1}	0	∞	33
Inverse Mellin	y^{-x}	$c - i\infty$	$c \quad i\infty$	26
Hankel	$J_\nu(xy)(xy)^{\frac{1}{2}}$	0	∞	88
Y-transform	$Y_\nu(xy)(xy)^{\frac{1}{2}}$	0	∞	25
K-transform	$K_\nu(xy)(xy)^{\frac{1}{2}}$	0	∞	29
H-transform	$\mathbf{H}_\nu(xy)(xy)^{\frac{1}{2}}$	0	∞	16
Kontorovich-Lebedev	$K_{ix}(y)$	0	∞	3
Riemann-Liouville fractional integral	$\dfrac{1}{\Gamma(\mu)}(y - x)^{\mu-1}$	0	y	16
Weyl fractional integral	$\dfrac{1}{\Gamma(\mu)}(x - y)^{\mu-1}$	y	∞	12
Stieltjes	$\dfrac{1}{(x + y)^\rho}$	0	∞	23
Hilbert	$\dfrac{1}{x - y}$	$-\infty$	∞	20

For each transform the functions $f(x)$ are arranged according to a certain hierarchy of functions which is explained in the preface to volume I. These integral transforms fill the whole of volume I and about three-fifths of volume II. The remaining part of volume II is taken up by miscellaneous definite integrals which cannot be conveniently written as an integral transform of one of the above types.

Only for a few of the better-known transforms (the first five in the above list) the authors have been able to avail themselves of significant previous collections of integrals. Most of the other material is either new or has been extracted from the periodical literature. Professor Oberhettinger contributed most of the miscellaneous integrals in volume II.

The following remarks concerning accuracy and completeness of the tables should be regarded in the light of the unquestionable merit and great competence of the work under review. Unfortunately there exists for tables of formulas no simple device for detecting errors such as differencing. As a consequence, few such tables can match the high standards of accuracy which are now maintained

by the best numerical tables, and also the table under review contains a number of misprints and minor errors. The reviewer has been informed that an incomplete list of corrigenda to volume I has been prepared and is available from the publisher. A more extensive list, covering both volumes, is in preparation and will be available at the same place. Turning to completeness, it cannot of course reasonably be expected that all integrals which have ever been evaluated can be found in the tables. One would expect, however, that for those integrals which are listed the results are given in complete form. There are some exceptions to this, particularly in the tables of Hankel transforms, where some of the given results do not cover the entire range of y.

The editors of *MTAC* have agreed to publish desirable future addenda to the tables. To this the reviewer would like to add a suggestion of his own. Very generally speaking, integral transforms are frequently met in problems with infinite domains. Problems with finite domains often are solved conveniently by series expansions. This suggests the usefulness of tables of series expansions involving higher transcendental functions. These tables could be arranged in much the same way as the present *Tables of Integral Transforms* and would form, in this reviewer's opinion, a welcome companion volume to the present set.

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108[S, X].—BERNARD FRIEDMAN, *Principles and Techniques of Applied Mathematics*, John Wiley & Sons, Inc., New York, 1956, ix + 315 p. Price $8.00.

Professor Friedman has written a most excellent and useful book which will go far in bridging the gap between the pure and applied mathematics of the spectral theory of ordinary and partial differential equations. In any book of an introductory nature of this size it is impossible of course to cover the range of applied mathematics in anything like the sense it is covered in Frank-von Mises, and so Friedman has wisely concentrated on providing the mathematical tools appropriate for scattering problems in acoustics, electromagnetic theory and in quantum mechanics. In the course of doing this he has at the same time covered a great deal of common ground essential for all of applied mathematics.

The book consists of five chapters entitled: I. Linear Spaces; II. Spectral Theory of Operators; III. Green's Functions; IV. Eigenvalue Problems of Ordinary Differential Equations; V. Partial Differential Equations. The pedagogy is excellent throughout. Friedman proceeds from the simplest example of a given phenomenon to a sufficiently general example of its occurrence to provide complete understanding of it. The reader is further helped by the fact that the more difficult mathematical theorems are proved in a series of appendices at the end of the appropriate chapter. Thus the main thread of any given argument is more readily followed since a better separation of principles from details is achieved.

Novel features not usually found in previous books on these topics include: the use of the bra and ket notation of Dirac; the introduction and systematic use and justification of the distributions of L. Schwartz; the extensive use of contour integration in the manner of the Watson transformation; the use of the concept

of the "approximate spectrum" for the determination of the continuous spectrum of a given differential operator; the treatment of separation of variables by using the inverse of two commutative operators; the method of Kato for improving the results of a Rayleigh-Ritz eigenvalue estimation; the introduction of the concept of impedance after Schelkunoff for the usual Sturm-Liouville problem and its relations to the concepts of transmission and reflection coefficients; the systematic treatment of Sturm-Liouville boundary value problems when the boundary conditions are either inhomogeneous or when the eigenvalue enters explicitly in them and many others too numerous to be enumerated here.

While the reviewer would have preferred a more extensive discussion of variational methods and of the meaning of the notions of limit point and limit circle for differential equations and their relationship to the usual abstract Hilbert space formulation of the spectral theory, he believes that Professor Friedman has written a thoroughly enjoyable and useful book which all active applied mathematicians should certainly have as part of their library, and which would certainly represent at the same time a suitable text-book for either a semester or a year course in applied mathematics. (By way of incidental criticism, the concepts of covariant and contravariant vectors are never defined although on page 22 their distinction is likened to that existing between basis and reciprocal basis vectors and on page 26 to that between row and column vectors.)

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109[S].—H. PREUSS, *Integraltafeln zur Quantenchemie*, Springer-Verlag, West Berlin, Germany, 1956, iv + 162 p., 27 cm. Price DM 39.

In the calculation of the binding energy and electronic structure of molecules and solids by quantum mechanics, it is necessary to calculate the integral of products of wave functions, each about one or another atomic center, multiplied by various direct or inverse powers of the distance r_a from one of the centers to an element of integration or the distance r_{12} between two elements of integration. For first-order approximations only one- and two-center integrals are needed, the effect on a bond between two atoms because of the presence of a third is neglected.

The tables under review give numerical values for the simpler pertinent one- and two-center integrals, for atomic wave functions appropriate for K and L shells (i.e., for constituent atoms up to neon in the periodic table) for a fairly wide range of the parameters involved. In addition, a number of subsidiary functions are tabulated, in terms of which the other pertinent functions may be computed. Thus approximate calculations of the structure and energy of such molecules as H_2O, NH_3, etc., may be worked out by the use of these tables plus a desk computer.

In the functions themselves, five significant figures are given, in the main; for the auxiliary functions six significant figures are given. This is adequate, since first-order calculations of binding energy often differ from measured values by factors of 2 or 3. In order to give a wide enough range of the parameters and

keep the tables reasonable in size, the intervals between tabulated values of the parameters are large enough so that interpolation would be difficult in many cases.

These limitations are not critical, however, if the limitations of the first-order computations, for which the tables are designed, are kept in mind. All that can be expected of first-order calculations is an order-of-magnitude estimate of energy and a crude estimate as to structure. It would therefore be illusory to interpolate parameter values, to try to get an accurate minimum, for example. To get accuracy sufficient to warrant interpolation, one would need to use at least second-order terms, which involve three-center integrals and other functions not tabulated here; such computations should probably be done on a high-speed computer.

For what they are supposed to do, the tables seem adequate. They are handsomely printed, with plenty of explanatory textual material and bibliographic references. The reviewer has not checked the accuracy of the entries.

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110[S].—BALTH. VAN DER POL, Editor, *Atlas of Ground-Wave Propagation Curves for Frequencies between 30 Mc/s and 300 Mc/s*, l'Union Internationale des Telecommunications, Geneva, 1955, xxxv + 174 p., 29.5 cm., oblong. Price 36.50 Swiss francs.

Curves are presented for the field of a half-wave transmitting dipole as a function of distance for a wide range of the physical parameters. These are: frequencies of 30, 60, 100, 150, 200, and 300 megacycles per second, heights of transmitting antenna above the ground of 10, 20, 50, 100, 200, 500, and 1000 meters, and heights of receiving antennas above the ground of 0, 2, 5, 10, 20, 50, 100, 200, 500, and 1000 meters. The transmitting and receiving antennas are oriented, simultaneously, horizontally, or vertically. The electrical ground constants are taken corresponding to sea water ($\sigma = 4$ mhos/m and $\epsilon/\epsilon_0 = 80$) and to land ($\sigma = 10^{-2}$ mhos/n and $\epsilon/\epsilon_0 = 10$). To account for "normal" atmospheric refraction, the earth's radius is taken as 4/3 times the actual earth's radius.

The r.m.s. field strength in micro-volts/meter is plotted against distance D from 0 to 200 kilometers for a specified antenna height, frequency and ground constants. The strength of the transmitter is such that the hypothetical field, in its equatorial plane, would be

$$E = \frac{2.22 \cdot 10^5}{D_{km}} \mu V/m$$

if the latter was placed in free space.

One of the underlying assumptions is: the earth is regarded as a smooth and homogeneous sphere. The inhomogeneous character of the atmosphere has not been considered. Effects such as "inversion layers" and turbulence are thus excluded. Furthermore, the roughness and variability of the ground constants are not mentioned. It is unfortunate that a few calculations were not carried out for

an earth conductivity of 10^{-3} mhos per meter corresponding to poorly conducting ground which is more typical of arid regions of the world.

As a result of the simplifying assumptions mentioned above, the problem is reduced to a scalar boundary value problem in electromagnetic theory. Essentially it is desired to obtain a Green's function G exterior to a spherical region with an approximate boundary condition $\partial G/\partial r = \gamma G$ at the surface. The constant γ depends on the electrical constants of the ground, the polarization and the frequency. The validity of this boundary condition rests on the condition that the square of the refractive index N^2 of the earth is large compared to one. Some error might be introduced in employing values of γ based on this assumption when $|N^2|$ is only of the order of 10 for over-land propagation at 30 Mc/s.

For purposes of computation, G is represented as a sum of modes involving spherical Bessel Functions of large complex order with a large real argument. These Bessel functions are approximated by using their representation in terms of Hankel functions of order 1/3 or the equivalent modified Airey Integrals. This Hankel approximation has been discussed extensively in the early work of van der Pol and Bremmer referred to in the text. The statement on page XIII that "the introduction of the Hankel approximation is essentially equivalent to the use of an approximate boundary condition . . ." is, however, not true. These are two separate approximations with different regions of validity.

When the transmitting and receiving antennas are at moderate or very large heights, geometrical optics provide a much simpler formula for computation. Most of the curves in the atlas are based both on the geometrical-optical and the spherical wave function formulas. The overlapping of the regions of usefulness of these two techniques provides an excellent check on the over-all accuracy.

Most of the calculations were carried out under the supervision of Professor A. Van Wijngaarden on the electronic digital calculator (ARRA) of the Mathematical Centre in Amsterdam. Parts of the text were written by Dr. H. Bremmer.

The curves provide a convenient reference for field strength estimates in the frequency range from 30 to 300 Mc/s when the atmosphere is undisturbed and inversion layers are not present.

The list of errata below has been compiled since the publication of this book and furnished by Professor van der Pol. The corrections apply only to the introductory text of the atlas and in no way influence the validity of the curves.

page	for	read
VI, lines −11 and −12	on a perfectly conducting plane	in free space
IX, line 9	1.47	1.047
IX, line 13	having the same moment	radiating 1 kilowatt
X, line −11	azimuthal	angular
XIV, formula (7) XV, formula (10)	787 000	$\dfrac{787\ 000}{D_{km}}$

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111[S].—J. M. KENNEDY & M. J. CLIFF, *Transformation Coefficients between LS and jj Coupling*, Atomic Energy of Canada, Limited, Chalk River, Ontario, A.E.C.L. Report No. 224, 1955, 39 p., 27 cm. Price $0.50.

This report lists and discusses the transformation coefficients for converting wave functions expressed in the *LS* coupling representations to a *jj* representation or vice versa. The basic relation used is:

$$\psi(l_1 l_2 L, s_1 s_2 S, JM) = \sum_{j_1 j_2} A \begin{pmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{pmatrix} \psi(l_1 s_1 j_1, l_2 s_2 j_2, JM),$$

where l_1 and l_2 are orbital angular momenta whose resultant is L; s_1 and s_2 are spins whose resultant is S; j_1, the total angular momentum of particle i, is the resultant of l_1 and s_1, and J, the total angular momentum of the two particle system, is the resultant of L and S (and also of j_1 and j_2). The expansion coefficients A are independent of M the z component of the angular momentum J. In atomic and nuclear spectroscopy, the spins s_1 and s_2 are equal to $\frac{1}{2}$. In the report, the coefficients,

$$A \begin{pmatrix} l_1 & \frac{1}{2} & j_1 \\ l_2 & \frac{1}{2} & j_2 \\ L & S & J \end{pmatrix}$$

are given in three different forms:

1. as algebraic functions of l_1, l_2, J,
2. numerically (up to 8 decimal places) for all possible cases in which l_1 and l_2 do not exceed 5,
3. in coded fractional form for all cases in which l_1 and l_2 do not exceed 5. The coded fractional form is useful since the squares of the A coefficients are rational fractions which have no prime factors exceeding the value 19 for the ranges of parameters considered here.

We give here an example of this method—it is discussed more fully in the report.

$$18/275 = 2' \times 3^2 \times 5^{-2} \times 7^0 \times 11^{-1} \rightarrow 1220, 1,$$

the underlining denoting a negative exponent and a comma being placed after the fourth digit. Multiplication of fractions merely involves addition of their fractional representations.

The tables are well presented (and legible) and should be useful to workers in atomic and nuclear spectroscopy.

Related tables are:

W. T. SHARP, J. M. KENNEDY, B. J. SEARS, & M. G. HOYLE, *Tables of Coefficients for Angular Distribution Analysis*, CRT-556, A.E.C.L. Report No. 97 [*MTAC*, Rev. **26**, v. 10, 1956, p. 51–52].

J. M. KENNEDY, B. J. SEARS, & W. T. SHARP, *Tables of X Coefficients*, CRT-569, A.E.C.L. Report No. 106 [*MTAC*, Rev. **27**, v. 10, 1956, p. 52].

S. A. MOSZKOWSKI

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112[C, K, L, W, Z].—Robert R. Bush & Frederick Mosteller, *Stochastic Models for Learning*, John Wiley & Sons, Inc., New York, 1955. Price $9.00.

This work is of interest to readers of *Mathematical Tables and Other Aids to Computation* both for the text and for tables which are included as incidental information to provide means for evaluation of the validity of hypotheses made or suggested in the text.

Major tables include the functions: $\Phi(\alpha, \beta) = \sum_{r=0}^{\infty} \alpha^{r(r+1)/2}\beta^r$ and $\Psi(\alpha, \beta)$

$= \sum_{r=0}^{\infty} r\alpha^{(r+1)/2}\beta^r$, $\alpha = .5(.02).99$, $\beta = .5(.02)1$, 4D; $T(\alpha, \beta) = \dfrac{-\log\dfrac{1-\alpha}{1-\beta}}{\alpha - \beta}$ over

the range $\alpha, \beta = .7(.01).99$ with $\alpha \le \beta$, 3D; $g_r(\alpha) = \dfrac{r\alpha^r}{1 - \alpha^r}$, $\alpha = .50(.01).99$,

$\nu = 1(1)10$, 3D; $F(\alpha, \beta, \Omega) = \sum_{r=0}^{\Omega} \dfrac{\alpha^r\beta}{1 - \alpha^r\beta}$ and $G(\alpha, \beta, \Omega) = \sum_{r=0}^{\Omega} \dfrac{r\alpha^r\beta}{1 - \alpha^r\beta}$,

$\alpha, \beta = .5(.05).7(.02).98$, $\Omega = 4(4)16$.

In addition to these tables (which were considered by the authors to be sufficiently important to be presented separately as appendices) there are numerous tables scattered throughout the text. Some of these simply tabulate observed data from experiments reported, and others furnish functions which the authors need for their exposition but which they evidently consider to be of less general interest. An example of such a table is the function $P(\alpha, \beta) = \prod_{r=0}^{\infty} (1 - \alpha^r\beta)$, $\alpha, \beta = 0(.1)1$, 3D found on page 160.

The major tables are used in maximum likelihood estimates and in other adjustment of parameters in the theory developed.

The book also contains a general description of the mathematical and numerical methods used in making the estimates. These methods are not new, but the whole exposition seems to the reviewer to be a valuable contribution to literature generally applicable to the rapidly expanding quantitative social sciences.

The aim of the authors is to present a hypothesis with regard to the learning process. This is briefly that a reaction to a stimulus is probabilistic and that the governing probability distribution is dependent on the experience of the person subjected to the stimulus. A major portion of the book is devoted to an attempt to make quantitative hypotheses along these lines which accord with experience.

The tables are auxiliary to the estimates required in these hypotheses.

The theory itself is of plausible interest to students of automatic computing equipment. This interest might work in both directions—the equipment might give better simulations of the learning processes assumed and hence better insight into the theory, and the theory might be put to use in the design and application of some types of machinery, say, for language translation or for some similar application. However, the connection at present must certainly be considered to be nebulous.

The tables are a minor part of the book; they are a little hard to read because of the space saving device of listing two functions with a single table of arguments. Acknowledgment of help in computation is given, but no indication of checks of accuracy or of means used. There are no aids to interpolation.

C. B. T.

113[Z].—MARTIN H. WEIK, PB 111996, *A Survey of Domestic Electronic Digital Computing Systems* (Reprint of Ballistic Research Laboratory Report No. 971), United States Department of Commerce, Office of Technical Services, Washington, D. C., 1956, vii + 272 p., 28 cm. Price $4.75.

It is certainly not easy to remain abreast of developments in the electronic computer field. This survey helps to bring up-to-date (end of 1955) a broad picture of the field in the United States. Previous surveys of a similar nature date back to 1953 and are now obsolescent.

Eighty-four "Domestic Electronic Digital Computing Systems" are described in the report. Two foreign manufacturers, Ferranti and Olivetti, are represented, presumably because these companies have U. S. sales organizations. On the other hand at least some U. S. computers are missing, for example Harvard Mark IV, UNIVAC 120, and UNIVAC File Computer. It would be highly desirable if all foreign computers could be added to the list together with annual supplements.

The descriptive information on each system runs from 2 to 4 pages and generally includes photographs, manufacturer, user(s), arithmetic and logical organization, size, cooling and power, input-output and memory data, production record and price, and operating experience.

Much of the information was obtained from manufacturers and users. For this reason, as the editor points out, care must be taken when comparing detailed minute points since the contributers may have used different frames of reference. Anyone considering the acquisition of a computer could use the information in the survey to select a small group of computers for further consideration. The survey would be very useful for familiarization purposes.

A chapter on Analysis and Trends follows the system description. The machines are listed in a number of ordered tables according to the characteristics of word length, add time, memory capacity, memory access time, tube and diode quantities, power requirements and cost. In addition the editor gives his opinion of trends in the field.

A brief bibliography and a glossary of computer terms are included.

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TABLE ERRATA

Reviews and papers in this issue mention errata in the following works:

A. ASHOUR & A. SABRI, "Tabulation of the function $\psi(\theta) = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2}$," Review 105, p. 250.

ROBERT E. GREENWOOD, "Coupon collector's test for random digits," Review 96, p. 243.

H. HASSE, *Vorlesungen über Zahlentheorie*, p. 197.

F. G. TRICOMI, "Valori numerici di ortogonali di Laguerre," Review 103, p. 248.

BALTH. VAN DER POL, *Atlas of Ground-Wave Propagation Curves for Frequencies between 30 Mc/s and 300 Mc/s*, Review 110, p. 256.

249.—W. Müller, "Viscous flow within cylindrical boundaries," *Ann. d. Physik.* [*MTAC*, v. 1, 1944, p. 263.]

Items 7 and 9, *for* $J_0(rj_{1,n})$ *read* $J_1(rj_{1,n})$.

H. D. Conway

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250.—Herbert E. Salzer, "On numbers expressible as the sum of four tetrahedral numbers," London Math. Soc., *Jn.*, v. 20, 1945, p. 3–4.

The following erratum has been found:

Page 3, line 25, *the number 107 was omitted. It should be inserted between numbers 103 and 137.*

Herbert E. Salzer

CONVAIR
San Diego, California

251.—George Wellington Spenceley, Rheba Murray Spenceley, & Eugene Rhodes Epperson, *Smithsonian Logarithmic Tables to Base e and Base 10*, The Smithsonian Institution, Washington, D. C., 1952.

The following errata have been found.

p. 336, log 6686, first figure, *for* 6 *read* 8
p. 342, log 6997, third figure, *for* 9 *read* 4
p. 349, log 7347, fourth figure, *for* 0 *read* 1
p. 364, log 8100, first figure, *for* 8 *read* 9.

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NOTES

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Index

The classified index of reviews in this volume continues along the general lines set in volume 9 (see *MTAC*, v. 9, 1955, p. 226–229). However, in accord with suggestions received from many different people, an attempt has been made to describe the work reviewed rather than to list the title.

For tables in analysis, the description usually describes the functions tabulated, the increments and ranges of the arguments, the precision of the table. The description for other tables and works is less stereotyped, ranging from the fairly precise description above to vague statements about intended or presumed use of the work.

In the formal description the symbol δ^n is used to indicate that n-th differences of some kind—possibly modified with throw-back of a higher difference, for example—are included.

Tables from Number Theory are arranged more or less in the order of the classification scheme of D. H. Lehmer [1]. Tables from Analysis are arranged more or less according to the classification scheme of A. Fletcher, J. C. P. Miller, and L. Rosenhead [2], usually carried to 1D. Notation of these authors has been taken as standard wherever possible. Tables from Statistics are arranged more or less in the order of the classification scheme furnished last year by Professor H. O. Hartley. It has been convenient to add a few classes to these schemes.

The Chairman of the Editorial Committee accepts full responsibility for the mistakes or bad judgment displayed in the classified and other indices. He invites suggestions for future volumes. He also acknowledges gratefully great aid rendered by Professor W. J. Dixon, particularly in connection with Tables from Statistics.

Usually in *Mathematical Tables and Other Aids to Computation* ranges and increments of variables are expressed in the notation,

$$a_1(d_1)a_2(d_2)a_3(d_3)a_4,$$

etc., which means that the first value of the argument is a_1, the increment is d_1 between argument values a_1 and a_2 (inclusive), the increment is d_2 between a_2 and a_3, the increment is d_3 between a_3 and a_4, and (in this example) the last value of the argument is a_4.

C. B. T.

1. D. H. Lehmer, *Guide to Tables in the Theory of Numbers*, National Research Council, Washington, D. C., 1941.
2. A. Fletcher, J. C. P. Miller, & L. Rosenhead, *An Index of Mathematical Tables*, Scientific Computing Service Limited, London, 1946.

CORRIGENDA

Andrew D. Booth, *Numerical Methods*, *MTAC*, v. 10, 1956, Review 38, reference 3, p. 166, *for*

on p. 53 we find + etc., $\cdots$, $\cdots$ etc.

read

on p. 53 we find + etc. $\cdots$, $\cdots$ etc.

HARVEY COHN, "Stability configurations of electrons on a sphere," *MTAC*, v. 10, 1956, p. 117.

Föppl's results for $n = 8$ were misquoted. The configurations corresponding to the cube and dodecahedron are not stable.

Mr. John Leech has been kind enough to point this out and suggest the further, rather comprehensive reference: L. L. WHYTE, "Unique arrangements of points on a sphere," *Amer. Math. Monthly*, v. 59, 1952, p. 606–611.

W. MÜLLER, "Viscous flow within cylindrical boundaries," *Ann. d. Physik*, *MTAC*, v. 1, 1944, p. 263, items 7 and 9 should be corrected in accordance with Table Errata **249** this issue.

K. C. S. PILLAI & K. V. RAMACHANDRAN, "On the distribution of the ratio of the ith observation in an ordered sample from a normal population to an independent estimate of the standard deviation," *MTAC*, v. 10, 1956, Review 13, p. 43, line 8 from bottom, *for* v. 25, 1955, p. 565–572, *read* v. 25, 1954, p. 565–572; p. 44, line 3, *for*

$$e^{ikx^2/6} \left(\sum_{i=0}^{\infty} a_i^{(k)} x^i \right)$$

read

$$e^{ikx^2/6} \left(\sum_{i=0}^{\infty} a_i^{(k)} x^i \right).$$

THE RAND CORP., *One Million Random Digits and 100,000 Normal Deviates*, *MTAC*, v. 10, 1956, Review 11, p. 39, in the title,

for   *One* Million Random Digits . . .

read   *A* Million Random Digits . . . .

In connection with the number of pages,

for   xxv $+$ 200 p.

read   xxvii $+$ 400 $+$ i $+$ 200 p.

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